GLOBAL IDENTIFIABILITY UNDER UNCORRELATED RESIDUALS

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Global Identifiability under Uncorrelated Residuals

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Suppose in each equation, not counting covariance restrictions, we need one more restriction to meet the order condition. If we now add to each equation a restriction that its structural residual is uncorrelated with the residual of some other equation, is the parameter of the new model identifiable globally? That is the question.

In general the answer is no. The parameter could remain either not identifiable or is locally identifiable, possibly globally under additional inequality restrictions. In this paper we find families of models for which the answer to the question is yes without the help of inequalities.

The families share common characteristics. First, the sufficient condition for local identifiability must hold. Secondly, the string of zero correlations between residuals contains a closed cycle of length at least four. Thirdly, with the variables, equations and residuals all numbered as they are in the cycle, the odd numbered variables must satisfy a kinship relationship and lastly, the structural residuals can not all be uncorrelated. There are also differences in the families, but these come from the difference in the required kinship relationship.

When there are four or more equations containing external variables, the variety of models with uniquely identifiable parameter under a string of uncorrelated residuals is considerable. In particular, when correlated inverse demand shocks are uncorrelated with correlated supply shocks, our results show that many flexible inverse demand and supply equations reproducing exactly the observed price and quantity moments are members of the above families.

JEL Classification: C3
Keywords: unique identifiability, uncorrelated shocks, cyclical covariance restrictions, kinship, siblings, parental lists
1. Introduction.

Global identifiability under covariance restrictions is a delicate matter. For one reason, all equations have to be considered simultaneously and an equational perspective, as in Hausman & Taylor (1983) or in some parts of Bekker & Pollock (1986), reveals the links between the parts but not necessarily the wholeness of the system.

Consider the linear model \( By = u, E(u) = 0, E(u'u') = \Sigma = (\sigma_{ij}) \), where \( B = (\beta_{ij}), \beta_{ii} = -1, i, j = 1, ..., G \). Let \( Z_k \) be the kth row of \( Z \). The parameter \((B, \Sigma)\) of recursive models when \( B \) is lower triangular and \( \Sigma \) diagonal is globally identifiable because \((B_1, \sigma_{11})\) is identifiable without any covariance restriction, \((B_2, \sigma_{12}, \sigma_{22})\) is identifiable with the help of the restriction \( \sigma_{12} = 0 \) and the last equation \((B_G, \Sigma_G)\) is identifiable because of the \( G-1 \) restrictions on \( \Sigma_G \). Extensions to other cases of global identifiability under covariance restrictions as stated in Koopmans (1950) and in Theorem 4 of Wegge (1965) all have the characteristic that a first equation is identifiable without any covariance restrictions and that the restriction \( \sigma_{ij} = 0, i < j \), can be applied unambiguously to the identifiability of \((B_1, \sigma_{11}, ..., \sigma_{jj})\) given that \((B_1, \sigma_{11}, ..., \sigma_{ii})\) is identifiable already without it.

In an effort to expand on the scope of globally identifiable model parameters, Mallela and Patil (1976), Mallela (1989) and Mallela, P., Porter-Hudak S. and Yoo S-H. (1993) considered models and restrictions of the type

\[
B = \begin{pmatrix}
-1 & \beta_{12} & 0 & 0 \\
0 & -1 & \beta_{23} & 0 \\
\beta_{31} & 0 & -1 & 0 \\
0 & \beta_{42} & 0 & -1
\end{pmatrix}, \quad \Sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & 0 & 0 \\
\sigma_{21} & \sigma_{22} & 0 & 0 \\
0 & 0 & \sigma_{33} & \sigma_{34} \\
0 & 0 & \sigma_{43} & \sigma_{44}
\end{pmatrix},
\]

in which not a single equation is identifiable without covariance restrictions. We will call this the model \([2312] | \sigma_{13} = \sigma_{32} = \sigma_{24} = \sigma_{41} = 0\). They have obtained some fragmented results and their examples are the inspiration for the design of the more general models considered here.
2. The model \{ [j_1, \ldots, j_G] | \sigma_{m_i n_i} = 0, \ i = 1, \ldots, H, \ G \leq H \}.

Assume that in the $i$th row of $B$ with $\beta_{ii} = -1$, all its other elements are zero except for one unknown element $\beta_{i j_i}$, in column $j_i$, $i = 1, \ldots, G$. Call $j_i$ the parent of variable $i$. Including the normalization restriction there are only $G-1$ restrictions on $B_i$ and additional restrictions essential for identifiability are covariance restrictions. To this end, assume in the structural covariance matrix $\Sigma$ the elements $\sigma_{m_i n_i}, m_i z n_i$, $i = 1, \ldots, H, G \leq H$ are required to be zero.

In this paper we seek to find the special cases of the model \{ [j_1, \ldots, j_G] | \sigma_{m_i n_i} = 0, \ i = 1, \ldots, H, \ G \leq H \} that have a globally identifiable parameter $\alpha = ((\text{vec}B)', (\text{vec}\Sigma \setminus')')$, where $\Sigma \setminus$ are all elements of $\Sigma$ on or below the diagonal. The results are extended readily to the general model with external variables in which the parameter $(B, \Gamma, \Sigma)$ is required to satisfy $G-1$ restrictions on $(B, \Gamma)_i, i = 1, \ldots, G$.

The analytic formulation is simple enough. For $G \times G$ nonsingular $T$, the parameter $(T B, T \Sigma T')$ is equivalent to the parameter $(B, \Sigma)$ if and only if

$$T = I_G + (\tau_1 (B^{-1}) j_1', \tau_2 (B^{-1}) j_2', \ldots, \tau_G (B^{-1}) j_G')',$$

with $\tau = (\tau_1, \ldots, \tau_G)$ satisfying the set of bilinear equations $(T \Sigma T')_{m_i n_i} = 0$

i.e. $\tau$ is a solution of

$$0 = (j m_i, n_i) \tau m_i + (j n_i, m_i) \tau n_i + \omega m_i j n_i \tau m_i \tau n_i, \ m_i z n_i, \ i = 1, \ldots, H, \ (1)$$

where we often let $(m, n) \equiv (B^{-1})_{mn} \circ ((\Omega B')_{mn}$ and $\Omega \equiv B^{-1} \Sigma B^{-1} = (\omega_{ij})$, $i, j = 1, \ldots, G$, is the reduced form covariance matrix.

If $\tau$ satisfies (1) the alternative parameter $(T B, T \Sigma T')$ satisfies

$$(TB)_{ij} = \beta_{ij} + \delta_{j j_i} \tau_i, \quad (T \Sigma T')_{ij} = \sigma_{ij} + (j_i, j) \tau_i + (j_j, i) \tau_j + \omega_{j_i j_j} \tau_i \tau_j.$$
where $\delta_{ij}$ is the Kronecker delta. If (1) implies $\tau_i = 0$, $B_i$ is identifiable and if $\tau = 0$, the parameter $\alpha$ is identifiable.

In general bilinear equations (1) have multiple solutions and therefore only local identifiability properties are expected to hold. If $\psi(\beta, \sigma) \neq 0$, with $\beta = \text{vec}B$, $\sigma = \text{vec}\Sigma$, is the list of restrictions, the local identifiability condition is that the Jacobian matrix $J(\alpha)$ of the system of restrictions $\psi((\text{vec}TB)^\prime, (\text{vec}T\Sigma T^\prime)^\prime) = 0$ has rank $G^2$ at $T = I_G$. Or equivalently with $B$ nonsingular

$$J(\alpha)[I_G \otimes (B^\prime)^{-1}] \equiv [\psi \beta, (I_G \otimes B^\prime) + 2\psi \sigma, Q_G (I_G \otimes \Sigma)] [I_G \otimes (B^\prime)^{-1}]$$ (2)

has rank $G^2$. From (2), after deletion of rows and columns corresponding to the restrictions on $B$, the parameter $\alpha$ is locally identifiable if, and under constant rank conditions, only if the matrix of coefficients in the linear parts of the equations (1) has rank $G$. Local identifiability is equivalent to $\tau = 0$ is an isolated solution of (1).

Whereas local identifiability is necessary, global identifiability results of any generality have to be based on conditions under which the solution to the system of equations (1) can be shown by algebraic manipulations to be unique. This involves much more than knowing the rank of the Jacobian matrix.

Identifiability can also be stated as the condition that the population moment estimator $\{ (B^*, \Sigma^*) \} | B^* \Sigma^* B^* - \Sigma^* = 0$ satisfying the restrictions

$0 = \sigma^*_{m_i n_i}$ is unique i.e. the system

$$0 = \omega_{m_i n_i} - \omega_{j_i m_i} \beta^*_{m_i j_i} + \omega_{j_i m_i} \beta^*_{m_i j_i} n_i n_i$$

$$+ (j_i m_i, n_i) \tau^*_{m_i} + (j_i n_i, m_i) \tau^*_{n_i} + \omega_{j_i m_i} \tau^*_{m_i} \tau^*_{n_i}, \quad i = 1, \ldots, H,$$ (3)

where $\tau^*_m = \beta^*_{m_i j_i} - \beta_{m_i j_i}$ and $\tau^*_n = \beta^*_{n_i j_i} - \beta_{n_i j_i}$, has a unique solution $\tau^* = 0$. The system (1) defining alternative parameters through the linear transformation operation is identical to the system (3)
defining alternative values of the population moment estimator. Identifiability means \((B^*, \Sigma^*)\) is unique and consistent.

In (2) and in the analysis below the elements of the matrix \(B^{-1} \Sigma \circ \text{RB'} \circ E(yu')\) play a crucial role. This is the matrix of covariances between variables and residuals. If \((i,j) = (B^{-1} \Sigma)_{ij} = B_j' \Omega_1 \circ E(y_i u_j) = 0\), the variable \(y_i\) is an exogenous or instruments variable with respect to the jth equation, or the jth residual is an instrument in the ith equation. The latter interpretation is developed in Hausman and Taylor (1983) in the context of an equational analysis. In recursive models with \(B\) lower triangular, \(B^{-1} \Sigma\) is lower triangular with \(E(y_i u_j) = 0\), \(i<j\), or variable \(i\) is exogenous in all equations \(j>i\). In the type of models considered here, non-exogeneity is the rule and exogeneity relations are the exception.


To a single covariance restriction \(\sigma_{k_1k_2} = 0\) corresponds in (1) an equation containing the two unknowns \((\tau_{k_1}, \tau_{k_2})\) and corresponding to two restrictions \(\sigma_{k_1k_2} = \sigma_{k_3k_4} = 0\) we have two equations containing three or four unknowns. To three or \(G_1\) restrictions of the type

\[
\sigma_{k_1k_2} = \sigma_{k_2k_3} = \ldots = \sigma_{k_{G_1}k_1} = 0, \quad 3 \leq G_1 \leq G,
\]

corresponds in (1) a subsystem of \(G_1\) equations in \(G_1\) unknowns \(\tau(G_1) \circ (\tau_{k_1}, \tau_{k_2}, \ldots, \tau_{k_{G_1}})\).

Definitions:

1. With \(G\) the number of equations, the \(H\) restrictions

\[
\sigma_{m_1n_1} = \ldots = \sigma_{m_in_i} = \ldots = \sigma_{m_Hn_H} = 0, \quad m_i \neq n_i, \quad i = 1, \ldots, H,
\]

a) are **adequate** if \(H \geq G\) and for each integer \(j\) there exists a different pair of subscripts \((m_i, n_i)\), with \(j \in (m_i, n_i), \quad j = 1, \ldots, G\).

b) are **connected** if adequate and \(m_i \in \{m_1, \ldots, m_{i-1}, n_1, \ldots, n_{i-1}\}, \quad i = 2, \ldots, H\).

c) are disjoint if \(\{m_i, n_i \mid i = 1, \ldots, G_1\} \cap \{m_i, n_i \mid i = G_1+1, \ldots, H\}\) is empty. \(G_1 < G\).
2. A $G_1$-cycle is a set of $G_1$ restrictions

$$c(G_1) \equiv \sigma_{k_1k_2} = \sigma_{k_2k_3} = \ldots = \sigma_{k_{G_1}k_1} = 0,$$

with $k_i$, $i=2,\ldots,k_G$, distinct.

3. A $G_1$-cyclical $G$-tuple is the set of $G$ restrictions

$$c(G_1)=0, \sigma_{G_1+1} = \ldots = \sigma_G = 0, \quad G_1 \leq G.$$

A disjoint $G$-tuple of covariance restrictions could be adequate, but is not connected. As is clear from (1), if $(j_{ni},m_i) \neq 0$ i.e. if the parent of variable $n_i$ is not exogenous in the $m_i$th equation, connected restrictions have the property that the $n_i$-th equation is identifiable if the equations $\{m_1,\ldots,m_i-1, n_1,\ldots,n_{i-1}\}$ are identifiable.

Examples of $G_1$-cyclical $G$-tuples are rectangular $G$-tuples which are systems of $G$ covariance restrictions containing the 4-cycle $\sigma_{k_1k_2} = \sigma_{k_2k_3} = \sigma_{k_3k_4} = \sigma_{k_4k_1} = 0$. Every 4-cycle can be represented as a rectangle in the covariance matrix $\Sigma$ by locating some $\sigma_{k_{i+1}}$ at $\sigma_{k_{i+1}k_i}$ if necessary. Similarly 3-cycles, 5-cycles and 6-cycles can be represented as triangles, pentagons and sexagons in $\Sigma$.

A connected 4-cyclical 6-tuple of restrictions

$$\sigma_{14}=\sigma_{43}=\sigma_{35}=\sigma_{51}=\sigma_{12}=\sigma_{56}=0$$

A 6-cycle of covariance restrictions

$$\sigma_{12}=\sigma_{23}=\sigma_{34}=\sigma_{45}=\sigma_{56}=\sigma_{61}=0$$
Our main result concerns the identifiability under a connected $G_1$-cyclical $G$-tuple of zero correlations. In considering the total number of $G_1$-cyclical $G$-tuples, the order in which the covariance restrictions are written does not matter. Their numbers for values of $G=4,5,6,\ldots,G$ are stated in Table 1 where $G\# = G(G-1)/2$ is the number of distinct off-diagonal elements in $G \times G$.

Table 1. Total Number of $G_1$-cyclical $G$-tuples in $G \times G$.

<table>
<thead>
<tr>
<th>Connected</th>
<th>G-tuple</th>
<th>4-tuple</th>
<th>5-tuple</th>
<th>6-tuple</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-cyclical</td>
<td>$G!G^6-4/2(G-3)!$</td>
<td>12</td>
<td>150</td>
<td>2160</td>
</tr>
<tr>
<td>4-cyclical</td>
<td>$G!G^5-5/2(G-4)!$</td>
<td>3</td>
<td>60</td>
<td>1080</td>
</tr>
<tr>
<td>5-cyclical</td>
<td>$G!G^6-6/2(G-5)!$</td>
<td>0</td>
<td>12</td>
<td>360</td>
</tr>
<tr>
<td>6-cyclical</td>
<td>$G!G^6-7/2(G-6)!$</td>
<td>0</td>
<td>0</td>
<td>60</td>
</tr>
<tr>
<td>G-cyclical</td>
<td>$(G-1)!/2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Not connected</td>
<td>$G!/(G!G#-G)!$</td>
<td>0</td>
<td>30</td>
<td>1345</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>15</td>
<td>252</td>
<td>5005</td>
</tr>
</tbody>
</table>

To see this, the total number of $G$-tuples is the number of different tuples of $G$ elements that can be selected from $G\#$ off-diagonal elements of $\Sigma$. Of these $(G-1)!/2$ are G-cycles since in the G-cycle

$$\sigma_1k_2 = \sigma_2k_3 = \ldots = \sigma_{kG1} = 0,$$

$k_i$ in turn can be selected in $G-i+1$ different ways and reading the G-cycle backwards is the same G-cycle. When $G=3$ there is one 3-cycle, namely $\sigma_{12}=\sigma_{23}=\sigma_{31}=0$.

As shown in Appendix 1 in $G \times G \Sigma$ the total number of $G_1$-cycles is equal to $G!/2G_1(G-G_1)!$ and each $G_1$-cycle can be embedded in $G_1G^{G-G_1-1}$ connected G-tuples. The product $G!G^{G-G_1-1}/2(G-G_1)!$ is the number of connected $G_1$-cyclical G-tuples. Riordan (1958) studies cycles and related constructions.
Among the 1345 not connected 6-tuples, 100 have all 6 integers present in their subscripts. Of the latter, ten are two disjoint triangles and ninety contain a rectangle, one element doubly joint and one disjoint element. The results of Theorem 1 below apply to the former but not to the latter. 1245 of the not connected 6-tuples have missing integers in their subscripts.

With these preliminaries the main results are now stated. Lemma 1 states that under local identifiability conditions and a \( G_1 \)-cycle of uncorrelated residuals, a solution \( z(G_1) \neq 0 \) of (1) implies each component of \( z(G_1) \) is not zero. It is proved in Appendix 2.

4. Lemma 1.

Consider system (1) consisting of the \( G_1 \) cyclical covariance restrictions

\[
(T \Sigma T')k_1k_2 = (T \Sigma T')k_2k_3 = \ldots = (T \Sigma T')k_{G_1}k_1 = 0.
\]

and either
\[
(j_{k_1}, k_2)(j_{k_2}, k_3) \ldots (j_{k_{G_1}}, k_1) \neq 0
\]

or
\[
(j_{k_2}, k_1)(j_{k_3}, k_2) \ldots (j_{k_1}, k_{G_1}) \neq 0.
\]

If \( z(G_1) = (z_{k_1}, z_{k_2}, \ldots, z_{k_{G_1}}) \neq 0 \) is a solution of (1), \( z_{k_1}z_{k_2}\ldots z_{k_{G_1}} \neq 0 \).

This means that if the model \([j_1 \ldots j_G], \sigma_{m_{i_1}n_i} = 0, i = 1, \ldots, H]\) contains the \( G_1 \)-cycle \( \sigma_{k_1k_2} = \ldots = \sigma_{k_{G_1}k_1} = 0 \), the equations \( (k_1, k_2, \ldots, k_{G_1}) \) are either all identifiable or none is identifiable when either the parent of variable \( k_i \) is not exogenous in the \( k_{i+1} \)-th equation or the parent of variable \( k_{i+1} \) is not exogenous in the \( k_i \)-th equation, \( i = 1, \ldots, G_1 \). Our main result is stated as Theorem 1 and it is proved in Appendix 4.
**Theorem 1.**

Given the model \( \{j_{1},...,j_{G},j_{G_1+1},...,j_{G}\}, \sigma_{m_i n_i} = 0, i=1,..G \) assume:

(A.1) The covariance restrictions are a connected \( G_1 \)-cyclical \( G \)-tuple with \( G_1 \) even, \( 4 \leq G_1 \leq G \) and its first \( G_1 \) restrictions are the \( G_1 \)-cycle

\[
\sigma_{k_1 k_2} = \sigma_{k_2 k_3} = \ldots = \sigma_{k_{G_1-1} k_{G_1}} = \sigma_{k_{G_1} k_1} = 0.
\]

(A.2) \( (B^{-1} \Sigma)_{j_{n_i} m_i} \neq 0, i=G_1+1,...,G. \)

(A.3) In \( \{1,2,...,G_1\} \) for some integer \( i, i \) even, and for all \( l, l \) odd, one of the following holds:

a) \( j_{k_l} = k_i. \)

b) \( j_{k_l} = k_{i+1}, l \neq i+1. \)

c) \( j_{k_l} = k_n, l \neq i+1, n \) even, \( n \neq \{i,i+2\} \) satisfies

\[
j_{k_n} = j_{k_{i+1}}, \quad \sigma_{k_i k_n} = \sigma_{k_{i+2} k_n} = 0, \quad j_{k_i} = j_{k_{i+2}}.
\]

d) \( j_{k_l} = j_{k_n}, l \neq i+1, n \neq \{i,i+1,i+2\} \) satisfies

\[
k_n = j_{k_{i+1}}, \quad \sigma_{k_i k_n} = \sigma_{k_{i+2} k_n} = 0, \text{ and}
\]

either \( j_{k_i} = j_{k_{i+2}}, \) or \( k = j_{k_{i+2}}, \) or \( k_{i+2} = j_{k_i}. \)

The parameter \( \alpha = ((\text{vec} \Sigma \cdot \Sigma)^{\prime})^{\prime} \) is globally identifiable if and only if \( \Delta(G_1) \equiv z_1(G_1) - z_2(G_1) \neq 0, \) where cyclically \( G_1+m \) stands for \( m \) and

\[
z_1(G_1) \equiv \prod_{l=1}^{G_1} a_l, \quad z_2(G_1) \equiv \prod_{l=1}^{G_1} b_l, \quad a_l \equiv (B^{-1} \Sigma), \quad b_l \equiv (B^{-1} \Sigma), \quad j_{k_l} k_{l+1} \quad j_{k_{l+1}} k_l.
\]

In the proof the \( G_1 \) bilinear equations are reduced to \( G_1/2 \) linear homogeneous equations. Uniqueness of the solution \( z=0 \) then follows from standard rank conditions \( \Delta(G_1) \neq 0, \) (A.3) are the reduction permitting assumptions in the \( G_1 \)-cycle.
5. Interpretation and implementation remarks.

1. The verbal understanding of (A.3) is that in a $G_1$-cycle of uncorrelated equations, the odd numbered variables are siblings. Variable $k_{i+1}$ where $i$ is an even number, is a possible exception. The common parent is
   a) $k_i$ under (A.3) a), with no exception,
   b) $k_{i+1}$ under (A.3) b), $k_{i+1}$ itself is the exception,
   c) $k_n$, a sibling of $k_{i+1}$, under (A.3) c), where $k_n$ is an odd numbered variable that is the exception and provided the adjacent variables $k_i$ and $k_{i+2}$ are siblings with residuals that are uncorrelated with the residual of the common parent $k_n$,
   d) $k_n$, a grandparent of $k_{i+1}$, under (A.3) d), where $k_n$ is an odd numbered variable that is the exception and provided the adjacent variables $k_i$ and $k_{i+2}$ are either siblings or direct descendants of each other with residuals that are uncorrelated with the residual of the common parent $k_n$.

   Since a cycle can be traversed forwards or backwards, in Theorem 1 the variable $k_{i+1}$ and its neighbors $k_i$ and $k_{i+2}$ could be replaced by $k_{i-1}$ and its neighbors $k_i$ and $k_{i-2}$.

   Clearly under (A.3) a), c) and d) at least one variable is parent to at least two variables. As we will show next, this is also implied by (A.3) b) and local identifiability. Therefore under Theorem 1 at least one variable is not a parent i.e. $B$ containing at least one unit column is reducible.

2. The inequality conditions (A.2) $(B^{-1} \Sigma)_{j \in \mathbb{N}_1} \neq 0$, $i = G_1 + 1, \ldots, G$, and $\Delta(G_1) \neq 0$ are equivalent to the local identifiability conditions (2). Define

   \[
   \delta_1 \equiv \begin{vmatrix} (B^{-1} \Sigma)_{j \in k_2} & (B^{-1} \Sigma)_{j \in k_3} \\ (B^{-1} \Sigma)_{j \in k_4} & (B^{-1} \Sigma)_{j \in k_3} \end{vmatrix}, \quad \delta_2 \equiv \begin{vmatrix} (B^{-1} \Sigma)_{j \in k_1} & (B^{-1} \Sigma)_{j \in k_4} \\ (B^{-1} \Sigma)_{j \in k_3} & (B^{-1} \Sigma)_{j \in k_4} \end{vmatrix}
   \]
Under (A.3) b) with either $k_3=j_{k1}$ or $k_1=j_{k3}$, (P.4) of Appendix 3 implies $\delta_2=0$ and $\Delta(4) = -(B^{-1}\Sigma)_{jk_1,k_2}(B^{-1}\Sigma)_{jk_3,k_4} \delta_1$. The condition $\Delta(4) \neq 0$ requires $\delta_1 \neq 0$ and therefore we must have $j_{k2}=j_{k4}$, $k_2=j_{k_4}$, $k_4=j_{k_2}$. Therefore, when $k_3=j_{k_1}$, the same variable $k_3$ must also be either the parent of $k_2$ or of $k_4$. One of the latter two variables could be the parent of $k_3$ but then the other has no descendant and the assumption (A.3) b) together with $\Delta(G_1) \neq 0$ also imply that $B$ is reducible, having at least one unit column.

If both $B$ and $\Sigma$ are conformably reducible, $B^{-1}\Sigma$ contains null submatrices and the local identifiability condition would fail. In particular, the parameter of a model satisfying (A.3) of Theorem 1 is not locally identifiable when $\Sigma$ is diagonal or also for $G=4$, when $B$ has two columns that are unit vectors. For larger systems $B$ may contain more unit vectors provided enough off-diagonal elements in $\Sigma$ are different from zero so that $B^{-1}\Sigma$ does not contain null submatrices. More precisely and operationally speaking the local identifiability condition (2) has to be verified.

We now list the models with globally identifiable parameter defined in Theorem 1 when $G=4$, followed by $G=5$ and $G=6$.

6. Four equation models with identifiable parameter.

There are three 4-cycles when $G=4$. These are the restriction systems

$S_1(4) \equiv \sigma_{13}=\sigma_{32}=\sigma_{24}=\sigma_{41}=0,$
$S_2(4) \equiv \sigma_{12}=\sigma_{23}=\sigma_{34}=\sigma_{41}=0,$
$S_3(4) \equiv \sigma_{12}=\sigma_{24}=\sigma_{43}=\sigma_{31}=0.$

Let $[j_1,j_2,j_3,j_4]$ be the list of parents i.e. the column indices of the non-zero unknown elements in the rows $(1,2,3,4)$ of $B$. There are 81 different sets of parents for each restriction system. The parameter is globally identifiable in 24 cases under (A.3) a) and in 48 cases under (A.3) b). These cases are the following.
a). 24 Globally Identifiable Cases: One variable is not a parent.

Let $F_a(S_i(4))$ be the family of models with $B$ nonsingular, $\Sigma$ positive definite and globally identifiable parameter $\alpha$ under the covariance restrictions $S_i(4)$ and (A.3) a) of Theorem 1. We have

$$F_a(S_1(4)) = \begin{pmatrix} [3312] & [3321] & [4412] & [4421] & \text{if } \sigma_{34} \neq 0 \\ [3422] & [4322] & [3411] & [4311] & \text{if } \sigma_{12} \neq 0 \end{pmatrix}.$$  

$$F_a(S_2(4)) = \begin{pmatrix} [2123] & [2321] & [4143] & [4341] & \text{if } \sigma_{24} \neq 0 \\ [2343] & [4323] & [2141] & [4121] & \text{if } \sigma_{13} \neq 0 \end{pmatrix}.$$  

$$F_a(S_3(4)) = \begin{pmatrix} [2142] & [2412] & [3143] & [3413] & \text{if } \sigma_{23} \neq 0 \\ [2443] & [3442] & [2113] & [3112] & \text{if } \sigma_{14} \neq 0 \end{pmatrix}.$$  

Each family has eight members satisfying (A.3) a) and the local identifiability condition $\Delta(4) \neq 0$. There are two members in each of four groups:

i) $j_k = j_{k_3} = k_2$, $j_{k_2} = j_{k_4}$, $j_{k_2} = j_{k_4}$, $j_{k_4} = j_{k_2}$. $|\Delta| = \sigma_{k_2 k_2} \sigma_{k_2 k_4} \rho_{k_1 k_3}$ $|B| = 3 \neq 0$

ii) $j_k = j_{k_3} = k_4$, $j_{k_2} = j_{k_4}$, $j_{k_2} = j_{k_4}$, $j_{k_4} = j_{k_2}$. $|A| = \sigma_{k_4 k_7} \sigma_{k_4 k_2} \rho_{k_3 k_1}$ $|B| = 3 \neq 0$

iii) $j_{k_2} = j_{k_3} = k_3$, $j_{k_3} = j_{k_1}$, $j_{k_3} = j_{k_1}$, $j_{k_1} = j_{k_3}$. $|A| = \sigma_{k_3 k_3} \sigma_{k_3 k_1} \rho_{k_2 k_4}$ $|B| = 3 \neq 0$

iv) $j_{k_2} = j_{k_3} = k_1$, $j_{k_3} = j_{k_1}$, $j_{k_3} = j_{k_1}$, $j_{k_1} = j_{k_3}$. $|A| = \sigma_{k_1 k_1} \sigma_{k_1 k_3} \rho_{k_2 k_4}$ $|B| = 3 \neq 0$

where $\rho_{k_1 k_2} = \sigma_{k_1 k_1} \sigma_{k_2 k_j}^{-1} \sigma_{k_1 k_j} \sigma_{k_1 k_2}$ and $|\Delta|$ is the absolute value of $\Delta(4)$.

The last two groups are obtained from the first two by rotating the subscripts of the 4-cycle one place i.e. by placing the first restriction last.

In each case the hypotheses that the unrestricted off-diagonal coefficients of $B$ or $\Sigma$ are zero, are testable. However the coefficients needed to keep $\Delta(4) \neq 0$ are not testable. In the graph below, $\sigma_{34} = 0$ is not testable.
The graph of Model \{[[3312]|S_1(4)]\}

The graph gives a representation of the local identifiability conditions i.e. the residuals \(u_3\) and \(u_4\) must be correlated. The assumption \((A.3)\) a) is shown by having \(y_1\) and \(y_2\) as siblings with \(y_3\) their parent, and \(y_4\) having no descendant.

b). 48 Globally Identifiable Case

Let \(F_b(S_1(4))\) be the family of models with \(B\) nonsingular, \(\Sigma\) positive definite and globally identifiable parameter \(\alpha\) under the covariance restrictions \(S_1(4)\) and \((A.3)\) b) of Theorem 1. We have

\[
F_b(S_1(4)) = \begin{cases} 
[2312] [2321] \text{ if } \beta_{23} \sigma_{34} \neq 0; & [2421] [2412] \text{ if } \beta_{24} \sigma_{43} \neq 0 \\
[3121] [3112] \text{ if } \beta_{13} \sigma_{34} \neq 0; & [4112] [4121] \text{ if } \beta_{14} \sigma_{43} \neq 0 \\
[3441] [4341] \text{ if } \beta_{41} \sigma_{12} \neq 0; & [3432] [3442] \text{ if } \beta_{42} \sigma_{21} \neq 0 \\
[4313] [3413] \text{ if } \beta_{31} \sigma_{12} \neq 0; & [3432] [4323] \text{ if } \beta_{32} \sigma_{21} \neq 0 
\end{cases}
\]

\[
F_b(S_2(4)) = \begin{cases} 
[3123] [3321] \text{ if } \beta_{32} \sigma_{24} \neq 0; & [3341] [3143] \text{ if } \beta_{34} \sigma_{42} \neq 0 \\
[2311] [2113] \text{ if } \beta_{12} \sigma_{24} \neq 0; & [4113] [4311] \text{ if } \beta_{14} \sigma_{42} \neq 0 \\
[2441] [4421] \text{ if } \beta_{41} \sigma_{13} \neq 0; & [4423] [2443] \text{ if } \beta_{43} \sigma_{31} \neq 0 \\
[4122] [2142] \text{ if } \beta_{21} \sigma_{13} \neq 0; & [2342] [4322] \text{ if } \beta_{23} \sigma_{31} \neq 0 
\end{cases}
\]

\[
F_b(S_3(4)) = \begin{cases} 
[4142] [4412] \text{ if } \beta_{42} \sigma_{23} \neq 0; & [4413] [4143] \text{ if } \beta_{43} \sigma_{32} \neq 0 \\
[2411] [2141] \text{ if } \beta_{12} \sigma_{23} \neq 0; & [3141] [3411] \text{ if } \beta_{13} \sigma_{32} \neq 0 \\
[2313] [3312] \text{ if } \beta_{31} \sigma_{14} \neq 0; & [2343] [3342] \text{ if } \beta_{34} \sigma_{41} \neq 0 \\
[3122] [2123] \text{ if } \beta_{21} \sigma_{14} \neq 0; & [2423] [3422] \text{ if } \beta_{24} \sigma_{41} \neq 0 
\end{cases}
\]
Each family has eight members satisfying (A.3) b) and the local identifiability condition $\Delta(4) \neq 0$. There are four members in each of four groups:

\[ \text{i) } j_1 = k_3, j_2 = j_4, j_4 = k_2, j_2 = k_1, j_1 = k_3, \quad |\Delta| = \beta_{k_3} \sigma_{j_3 j_3} \sigma_{k_2 k_4} \rho_{k_3 k_4} \quad |B| < 3 \neq 0 \]

\[ \text{ii) } j_3 = k_1, j_2 = j_4, j_2 = k_4, j_1 = k_4, j_2 = k_1, j_1 = k_3, \quad |\Delta| = \beta_{k_1} \sigma_{j_1 j_1} \sigma_{k_2 k_4} \rho_{k_3 k_4} \quad |B| < 3 \neq 0 \]

\[ \text{iii) } j_2 = k_4, j_3 = j_1, j_3 = k_1, j_1 = k_3, j_4 = k_2, \quad |\Delta| = \beta_{k_4} \sigma_{j_4 j_4} \sigma_{k_2 k_4} \rho_{k_3 k_4} \quad |B| < 3 \neq 0 \]

\[ \text{iv) } j_4 = k_2, j_4 = j_1, j_3 = k_1, j_1 = k_3, j_2 = k_4, \quad |\Delta| = \beta_{k_2} \sigma_{j_2 j_2} \sigma_{k_3 k_1} \rho_{k_4 k_2} \quad |B| < 3 \neq 0 \]

where $\rho_{k_1 k_2} = \sigma_{k_1 k_1} \sigma_{k_2 k_2} - \sigma_{k_1 k_2} \sigma_{k_2 k_1}$.

The elements of $(B, \Sigma)$ not required to be different from zero are testable. Thus in the model $\{[2312]\mid S_1(4)\}$ the restrictions $\beta_{12} = \beta_{31} = \beta_{42} = \sigma_{12} = 0$ or any subset are testable, but $\beta_{23} \sigma_{34} = 0$ is not.

In all cases one and only one of the four variables has no descendant and one has two. The twenty-four sets of parents listed under (A.3) a) reappear under (A.3) b) with a different 4-cycle of covariance restrictions. Graphically there are two types of models. Above we listed the cases with an example of one type followed by an example of the second type. In the first type, three variables stand in a triangular relationship and one of the three variables has the fourth variable as direct descendant. The four structural errors fall into two uncorrelated pairs. The second pair contains the fourth error that must be intracorrelated with that of its partner in the pair. This causes the latter's structural error to have an effect on the residual and on the parent of the fourth variable, where they collide. A graphical illustration with the correlated error $u_3$ and the fourth variable $y_4$ is this:
In the second type of models the triangular relation is replaced by a direct two-way relation between two of the variables. Again one variable is not a parent, but its residual and its grandparent are influenced by its partner's residual. A graphic illustration with $y_4$ as fourth variable and $u_1$ as its partner's residual is this:

**The graph of Model \{[2312] | S_1(4)\}**

![Diagram of Model \{[2312] | S_1(4)\}]

**The graph of Model \{[3312] | S_2(4)\}**

![Diagram of Model \{[3312] | S_2(4)\}]

c) With $G_1=4$, (A.3) c) is empty.

d). 24 Not Locally Identifiable Cases under any 4-cyclical restrictions

A. Under (A.3) d) when $i=2$, $n=1$ we have the group of cases having $k_1=j_{k_3}$, $j_{k_2}=j_{k_4}$ and therefore $\Delta(4)=0$. In this group are each one of the 12 cases with one variable having 3 descendants i.e.

$$[2111], [3111], [4111], [2122], [2322], [2422], [3313], [3323], [3343], [4441], [4442], [4443],$$

under each 4-cycle of restrictions after suitable permutation.
B. The 12 cases with two variables having two descendants are

\[
[2112], [2121], [3113], [2323], [2442], [3311], [3322], [3443], [4141], [4411], [4343], [4422].
\]

In all these cases \( \Delta(4) = 0 \), either because the model is equivalent to a model under \( S_1(4) \) with \( (B, \Sigma) \) conformably reducible satisfying

\[
k_m = j_{km+n} = j_{km+1+n}, \quad k_{m+2} = j_{km+2+n} = j_{km+3+n}, \quad m \in (1, 2), \; n \in (1, 2).
\]

or it satisfies for some value \( m \in (0, 1, 2, 3) \), either

\[
k_1 + m = j_{k2+m} = j_{k3+m}, \quad k_2 + m = j_{k1+m} = j_{k4+m}, \quad \text{or}
\]

\[
k_1 + m = j_{k2+m} = j_{k4+m}, \quad k_2 + m = j_{k1+m} = j_{k3+m}
\]

To see this, under the former conditions with \( m = 0 \), we have

\[
\begin{align*}
\Delta(4) &= (k_2, k_2)(k_1, k_3)(k_2, k_1)(k_1, k_4) - (k_1, k_2)(k_2, k_3)(k_1, k_1)(k_2, k_4) \\
&= (k_2, k_2)(k_1, k_3)(k_1, k_1)(k_2, k_4)(\beta_{k2k1} - \beta_{k1k2} = 0,
\end{align*}
\]

using the relation \( B_k(B^{-1}\Sigma)^{kj} \sigma_{kikj} - (k_i, k_j) + \beta_{k1k2} (j_{k1}, k_j) \).

The problem occurs with the imposition of vanishing covariance restrictions, not if covariances have known values that are not zero.

e). **Locally Identifiable** Cases: Each variable is a parent.

For completeness sake we record the remaining parental lists and cycles of zero correlations under which the parameter is locally identifiable. With each variable having one descendant, these models do not satisfy (A.3) and the parameter is not globally identifiable. These models are:

\[
\begin{align*}
[S_{3}(4)] & \quad \sigma_{12} = 0, \; \sigma_{34} = 0 \quad \text{are testable locally.} \\
[S_{4}(4)] & \quad \sigma_{13} = 0, \; \sigma_{24} = 0 \quad \text{are testable locally.} \\
[S_{3}(4)] & \quad \sigma_{13} = 0, \; \sigma_{24} = 0 \quad \text{are testable locally.}
\end{align*}
\]
An important application of Case [3412] is the inverse demand-supply system

\[
\begin{pmatrix}
-1 & 0 & \beta_{13} & 0 \\
0 & -1 & 0 & \beta_{24} \\
\beta_{31} & 0 & -1 & 0 \\
0 & \beta_{42} & 0 & -1
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
q_1 \\
q_2
\end{pmatrix}
= \begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{pmatrix}
\Sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & 0 & 0 \\
\sigma_{21} & \sigma_{22} & 0 & 0 \\
0 & 0 & \sigma_{33} & \sigma_{34} \\
0 & 0 & \sigma_{43} & \sigma_{44}
\end{pmatrix}.
\]

with

\[
\Delta = \left(\sigma_{33}\sigma_{44}\sigma_{12}^2-\sigma_{11}\sigma_{22}\sigma_{34}^2\right)/\left|B\right|^2, \quad \left|B\right| = (1-\beta_{13}\beta_{31})(1-\beta_{24}\beta_{42}).
\]

\[
\tau_1 = \frac{1-\beta_{13}\beta_{31}}{\beta_{31}}, \quad \tau_2 = \frac{1-\beta_{24}\beta_{42}}{\beta_{42}}, \quad \tau_3 = \frac{1-\beta_{13}\beta_{31}}{\beta_{13}}, \quad \tau_4 = \frac{1-\beta_{24}\beta_{42}}{\beta_{24}}.
\]

The parameter is locally identifiable if either \(\sigma_{12}\neq 0\) or \(\sigma_{34}\neq 0\). It is moment equivalent to one alternative parameter \(\alpha^*\) of the system

\[B^*y + \Gamma^*x = u^*,\]

with \(\beta_{13}^* = 1/\beta_{31}, \beta_{24}^* = 1/\beta_{42}, \beta_{31}^* = 1/\beta_{13}, \beta_{42}^* = 1/\beta_{24}.\) Under the inequality constraints \(\beta_{13} < 0, \beta_{24} < 0, \beta_{31} > 0, \beta_{42} > 0,\) this alternative parameter \((B^*, \Gamma^*, \Sigma^*)\) is not admissible. The parameter \(\alpha^*\) is not locally identifiable if \(\Sigma\) is diagonal. The assumptions on \(\Sigma\) are justified under the theory that inverse demand and supply disturbances are uncorrelated. Below we seek alternative specifications with identifiable parameter.
7. Five equation models with identifiable parameter.

There are sixty connected 4-cyclical quintuplets when \( G = 5 \), there being fifteen 4-cycles and each can have anyone of four different restrictions from the missing column (row) as fifth element. These are the restriction systems

\[
\begin{align*}
R_1(5) & \equiv \sigma_{13} = \sigma_{32} = \sigma_{24} = \sigma_{41} = \sigma_{m5} = 0, & R_{21}(5) & \equiv \sigma_{12} = \sigma_{23} = \sigma_{34} = \sigma_{41} = \sigma_{m5} = 0, \\
R_5(5) & \equiv \sigma_{13} = \sigma_{32} = \sigma_{25} = \sigma_{51} = \sigma_{m4} = 0, & R_{25}(5) & \equiv \sigma_{12} = \sigma_{23} = \sigma_{35} = \sigma_{51} = \sigma_{m4} = 0, \\
R_9(5) & \equiv \sigma_{13} = \sigma_{35} = \sigma_{54} = \sigma_{41} = \sigma_{m2} = 0, & R_{29}(5) & \equiv \sigma_{12} = \sigma_{25} = \sigma_{54} = \sigma_{41} = \sigma_{m3} = 0, \\
R_{13}(5) & \equiv \sigma_{15} = \sigma_{52} = \sigma_{24} = \sigma_{41} = \sigma_{m3} = 0, & R_{33}(5) & \equiv \sigma_{15} = \sigma_{53} = \sigma_{34} = \sigma_{41} = \sigma_{m2} = 0, \\
R_{17}(5) & \equiv \sigma_{53} = \sigma_{32} = \sigma_{24} = \sigma_{45} = \sigma_{m1} = 0, & R_{37}(5) & \equiv \sigma_{52} = \sigma_{23} = \sigma_{34} = \sigma_{45} = \sigma_{m1} = 0, \\
R_{41}(5) & \equiv \sigma_{12} = \sigma_{24} = \sigma_{43} = \sigma_{31} = \sigma_{m5} = 0, & R_{45}(5) & \equiv \sigma_{12} = \sigma_{24} = \sigma_{45} = \sigma_{51} = \sigma_{m3} = 0, \\
R_{49}(5) & \equiv \sigma_{12} = \sigma_{25} = \sigma_{53} = \sigma_{31} = \sigma_{m4} = 0, & R_{53}(5) & \equiv \sigma_{15} = \sigma_{54} = \sigma_{43} = \sigma_{31} = \sigma_{m2} = 0, \\
R_{57}(5) & \equiv \sigma_{52} = \sigma_{24} = \sigma_{43} = \sigma_{35} = \sigma_{m1} = 0, & R_{57}(5) & \equiv \sigma_{52} = \sigma_{24} = \sigma_{43} = \sigma_{35} = \sigma_{m1} = 0,
\end{align*}
\]

where \( \sigma_{mk} \) is a covariance with \( m \in \{1,\ldots,5\}, m \neq k \).

Any model with identifiable parameter when \( G = 4 \) could be augmented with a parent \( j_5 \) and a restriction \( \sigma_{m5} = 0, m \in \{1,\ldots,4\} \), to constitute a model with identifiable parameter when \( G = 5 \), provided \( (B^{-1} \Sigma)_{j5} \neq 0 \), \( j_5 \in \{1,\ldots,4\} \). This would hold under the local identifiability condition (2).

Examples of families of models \( F_a(R_k(5)) \) with \( B \) nonsingular and \( \Sigma \) positive definite having identifiable parameter under the covariance restrictions \( R_k(5) \) and (A.3) a) of Theorem 1 are

\[
\begin{align*}
F_a(R_1(5)) & = \begin{bmatrix} 3312 \lambda_1 & 3321 \lambda_2 & 4412 \lambda_3 & 4421 \lambda_4 \end{bmatrix} \text{ if } \sigma_{34} = 0, & \begin{bmatrix} 3422 \lambda_5 & 4322 \lambda_6 & 3411 \lambda_7 & 4311 \lambda_8 \end{bmatrix} \text{ if } \sigma_{12} = 0, & (B^{-1} \Sigma)_{\lambda_1} = 0, \\
F_a(R_{21}(5)) & = \begin{bmatrix} 2123 \mu_1 & 2321 \mu_2 & 4143 \mu_3 & 4341 \mu_4 \end{bmatrix} \text{ if } \sigma_{24} = 0, & \begin{bmatrix} 2343 \mu_5 & 4323 \mu_6 & 2141 \mu_7 & 4121 \mu_8 \end{bmatrix} \text{ if } \sigma_{13} = 0, & (B^{-1} \Sigma)_{\mu_1} = 0, \\
F_a(R_{41}(5)) & = \begin{bmatrix} 2142 v_1 & 2412 v_2 & 3143 v_3 & 3413 v_4 \end{bmatrix} \text{ if } \sigma_{23} = 0, & \begin{bmatrix} 2443 v_5 & 3442 v_6 & 2112 v_7 & 3112 v_8 \end{bmatrix} \text{ if } \sigma_{14} = 0, & (B^{-1} \Sigma)_{v_1} = 0, \\
F_a(R_{60}(5)) & = \begin{bmatrix} 0_1 \mu 422 & 0_2 4522 & 0_3 5433 & 0_4 4533 \end{bmatrix} \text{ if } \sigma_{23} = 0, & \begin{bmatrix} 0_5 4432 & 0_6 4423 & 0_7 5532 & 0_8 5523 \end{bmatrix} \text{ if } \sigma_{54} = 0, & (B^{-1} \Sigma)_{\mu_1} = 0.
\end{align*}
\]
where
\[
\begin{align*}
\lambda &\in \{1,2,3,4\}, \quad \mu_1,\mu_2,\mu_3,\mu_4, \quad \nu_1,\nu_2,\nu_3,\nu_4, \quad \omega_1,\omega_2,\omega_3,\omega_4
\end{align*}
\]
and \( F_a(R_{21}(5)) \) is obtained from \( F_a(R_1(5)) \) by interchanging rows and columns two and three, where \( F_a(R_{41}(5)) \) is obtained from \( F_a(R_{21}(5)) \) by interchanging rows and columns three and four and where \( F_a(R_{60}(5)) \) is obtained from \( F_a(R_{41}(5)) \) by interchanging rows and columns one and five.

Examples of families of models \( F_b(R_k(5)) \) with \( B \) nonsingular and \( \Sigma \) positive definite having identifiable parameter under the covariance restrictions \( R_k(5) \) and \( (A.3) \) of Theorem 1 are

\[
F_b(R_1(5)) = 
\begin{pmatrix}
[2312 \lambda_7] & [2321 \lambda_2] & [2421 \lambda_3] & [2412 \lambda_4] \\
[3121 \lambda_5] & [3112 \lambda_6] & [4112 \lambda_7] & [4121 \lambda_8] \\
[3441 \lambda_9] & [4341 \lambda_{10}] & [4342 \lambda_{11}] & [3442 \lambda_{12}] \\
[4313 \lambda_{13}] & [3413 \lambda_{14}] & [3423 \lambda_{15}] & [4323 \lambda_{16}]
\end{pmatrix}
\]

\[
F_b(R_{21}(5)) = 
\begin{pmatrix}
[3123 \mu_1] & [3321 \mu_2] & [3341 \mu_3] & [3143 \mu_4] \\
[2311 \mu_5] & [2113 \mu_6] & [4113 \mu_7] & [4311 \mu_8] \\
[2441 \mu_9] & [4421 \mu_{10}] & [4423 \mu_{11}] & [2443 \mu_{12}] \\
[4122 \mu_{13}] & [2142 \mu_{14}] & [2342 \mu_{15}] & [4322 \mu_{16}]
\end{pmatrix}
\]

\[
F_b(R_{41}(5)) = 
\begin{pmatrix}
[4142 \nu_1] & [4412 \nu_2] & [4413 \nu_3] & [4143 \nu_4] \\
[2313 \nu_9] & [3312 \nu_{10}] & [2343 \nu_{11}] & [3342 \nu_{12}] \\
[3122 \nu_{13}] & [2123 \nu_{14}] & [2423 \nu_{15}] & [3422 \nu_{16}]
\end{pmatrix}
\]

\[
F_b(R_{60}(5)) = 
\begin{pmatrix}
[0_15424] & [0_24524] & [0_34534] & [0_45434] \\
[0_54552] & [0_6452] & [0_75453] & [0_84553] \\
[0_93532] & [0_{10}3523] & [0_{11}3432] & [0_{12}3423] \\
[0_{13}5223] & [0_{14}5232] & [0_{15}4232] & [0_{16}4223]
\end{pmatrix}
\]

where
\[
\lambda_k \in \{1, 2, 3, 4\} \text{ with } (B^{-1} \Sigma) \lambda_k \neq 0, \mu_k \in \{1, 3, 2, 4\} \text{ with } (B^{-1} \Sigma) \mu_k \neq 0,
\]
\[
u_k \in \{1, 3, 4, 2\} \text{ with } (B^{-1} \Sigma) \nu_k \neq 0, \sigma_k \in \{5, 3, 4, 2\} \text{ with } (B^{-1} \Sigma) \sigma_k \neq 0, \quad k = \{1, \ldots, 12\}.
\]

With \( G = 5 \), Theorem 1 only contains results when the restrictions contain a 4-cycle. Pentagonal restrictions do not imply unique identifiability.

8. Six equation models with identifiable parameter.

When \( G = 6 \) we could have identifiability under anyone of the 1080 4-cyclical sixtuples. Anyone of the 45 4-cycles can be augmented with a pair of restrictions from 24 different possible pairs, each pair containing one element in each of the two columns not included in the 4-cycle. For example from \( S_1(4) \) we can construct the 24 connected sixtuples

\[
R_k(6) = \{ S_1(4) = \sigma_{l5} = \sigma_{m6} = 0, \ l \in \{1, \ldots, 4\}, \ m \in \{1, \ldots, 5\} \}, \quad k = 1, \ldots, 20,
\]
\[
R_k(6) = \{ S_1(4) = \sigma_{l6} = \sigma_{56} = 0, \ l \in \{1, \ldots, 4\} \}, \quad k = 21, \ldots, 24.
\]

The parameter of the model is identifiable under \( R_k(6), k = 1, \ldots, 20 \), under the conditions of Theorem 1 on \( \{j_1, j_2, j_3, j_4\} \) provided \( (B^{-1} \Sigma)_{j5} \neq 0 \) and \( (B^{-1} \Sigma)_{j6} \neq 0 \). Letting \( k = 1 \) when \( l = m = 1 \), examples of families of identifiable parameters are

\[
F_a(R(6)) = \begin{bmatrix}
3312 \lambda_1 \mu_1 & 3321 \lambda_2 \mu_2 & 4412 \lambda_3 \mu_3 & 4421 \lambda_4 \mu_4 & \text{if } \sigma_{34} \neq 0 \\
3422 \lambda_5 \mu_5 & 4322 \lambda_6 \mu_6 & 3411 \lambda_7 \mu_7 & 4311 \lambda_8 \mu_8 & \text{if } \sigma_{12} \neq 0
\end{bmatrix},
\]

\[
F_b(R(6)) = \begin{bmatrix}
2312 v_{10} & 2321 v_{20} & \text{if } \beta_{23} \sigma_{34} \neq 0; & 2421 v_{30} & 2412 v_{40} & \text{if } \beta_{24} \sigma_{43} \neq 0 \\
3121 v_{50} & 3112 v_{60} & \text{if } \beta_{13} \sigma_{34} \neq 0; & 4112 v_{70} & 4121 v_{80} & \text{if } \beta_{14} \sigma_{43} \neq 0 \\
3441 v_{90} & 4341 v_{100} & \text{if } \beta_{41} \sigma_{12} \neq 0; & 4342 v_{110} & 3442 v_{120} & \text{if } \beta_{42} \sigma_{21} \neq 0 \\
4313 v_{130} & 3413 v_{140} & \text{if } \beta_{31} \sigma_{12} \neq 0; & 3423 v_{150} & 4323 v_{160} & \text{if } \beta_{32} \sigma_{21} \neq 0
\end{bmatrix},
\]

with
\[ \lambda_n \in \{1, \ldots, 4\}, \ \mu_n \in \{1, \ldots, 4\}, \text{ with } (B^{-1}\Sigma)\lambda_n (B^{-1}\Sigma)\mu_n = 0, \ n = 1, \ldots, 8, \]
\[ \nu_n \in \{1, \ldots, 4\}, \ \omega_n \in \{1, \ldots, 4\}, \ \text{With } (B^{-1}\Sigma)\nu_n (B^{-1}\Sigma)\omega_n = 0, \ n = 1, \ldots, 16. \]

When \( G = 6 \) we could have identifiability also with a sexagon of restrictions of which there are 60. With \( k_i, i = 1, \ldots, 6, \) all distinct in the set \{1, \ldots, 6\}, define

\[ S_k(6) = \sigma_{k_1k_2} = \sigma_{k_2k_3} = \sigma_{k_3k_4} = \sigma_{k_4k_5} = \sigma_{k_5k_6} = \sigma_{k_6k_1} = 0. \]

From Theorem 1 we have the families of models \( [j_{k_1}, j_{k_2}, j_{k_3}, j_{k_4}, j_{k_5}, j_{k_6}] \) with identifiable parameters

\[ F_a(S_k(6), i) = \{ [k_1, j_{k_2}, j_{k_3}, j_{k_4}, k_i, j_{k_6}], \ (j_{k_2}, k_3)(j_{k_4}, k_5)(j_{k_6}, k_1) = (j_{k_2}, k_1)(j_{k_4}, k_3)(j_{k_6}, k_5), \ (k_i, k_2)(k_i, k_4)(k_i, k_6) = 0, \ i \text{ even} \} \]

\[ F_b(S_k(6), i = 2) = \{ [k_3, j_{k_2}, j_{k_3}, j_{k_4}, k_3, j_{k_6}], \ (k_3, k_2)(j_{k_2}, k_3)(j_{k_4}, k_5)(j_{k_6}, k_1) = (j_{k_2}, k_1)(j_{k_3}, k_2)(j_{k_4}, k_3)(j_{k_6}, k_5), \ (k_3, k_6) = 0 \} \]

\[ F_c(S_k(6), i = 2, n = 6) = \{ [k_6, j_{k_2}, j_{k_3}, j_{k_4}, k_6, j_{k_3}], \ (k_6, k_2)(j_{k_2}, k_3)(j_{k_4}, k_5)(j_{k_6}, k_1) = (j_{k_2}, k_1)(j_{k_3}, k_2)(k_6, k_4)(j_{k_6}, k_5), \ (j_{k_2}, k_3)(k_6, k_6) = 0, \ \sigma_{k_2k_6} = \sigma_{k_4k_6} = 0 \} \]

\[ F_d(S_k(6), i = 2, n = 1) = \{ [j_{k_1}, j_{k_2}, j_{k_1}, j_{k_2}, j_{k_1}, j_{k_6}], \ (j_{k_1}, k_2)(j_{k_1}, k_4)(j_{k_2}, k_5)(j_{k_6}, k_1) = (j_{k_2}, k_1)(j_{k_1}, k_3)(j_{k_6}, k_5), \ (j_{k_1}, k_6)(j_{k_2}, k_3) = 0, \ \sigma_{k_4k_6} = 0 \}. \]

For the 6-cycle \( S_1(6) = \sigma_{13} = \sigma_{32} = \sigma_{24} = \sigma_{45} = \sigma_{56} = \sigma_{61} = 0 \), examples with \( \Sigma \) positive definite and nonsingular \( B \) with the parents written in the order \([j_1, j_2, j_3, j_4, j_5, j_6] \) are

1) \([331231] \in F_a(S_1(6), 2), \text{ if } \sigma_{12}(\sigma_{25} + \beta_{23}\sigma_{35}) = \sigma_{22}(\sigma_{15} + \beta_{13}\sigma_{35}). \]

\[ |B| = 1 - \beta_{13}\beta_{31}. \]

2) \([231221] \in F_b(S_k(6), i = 2) \text{ if } \beta_{23}(\beta_{31}\sigma_{14} + \sigma_{34}) = 0, \ \sigma_{12}(\sigma_{25} + \beta_{23}\sigma_{35}) = \sigma_{22}\sigma_{15}. \]

\[ |B| = 1 - \beta_{12}\beta_{23}\beta_{31}. \]

3) \([632263] \in F_c(S_k(6), i = 2, n = 6) \text{ if } \beta_{63}\sigma_{21}\sigma_{34}\sigma_{35} = 0. \]

\[ |B| = 1 - \beta_{23}\beta_{32}, \ \sigma_{36} = \sigma_{46} = 0. \]

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4) \([315531] \in F_d(S_k(6), i=2, n=1)\) if \(\sigma_{52}\sigma_{36}\beta_{13}\sigma_{34}(\sigma_{33}+\beta_{35}\sigma_{53}) \neq 0, \sigma_{11}\sigma_{55}-\sigma_{15}\sigma_{51} \neq \sigma_{35}(\sigma_{51}\beta_{13}-\sigma_{11}\beta_{53}), |B| = 1-\sigma_{35}\sigma_{53}, \sigma_{14} = 0.\)

The above inequalities are the conditions that \(\Delta(6) \neq 0\) and correspond to the sufficient rank conditions (2) for local identifiability.

For general models their satisfaction almost everywhere can be checked by calculating \(B^{-1}\Sigma\) for some numerically chosen matrices \((B, \Sigma)\) that satisfy the equality restrictions. This is due to the fortunate fact that if all the coefficients in the system (1) are different from zero, the reciprocals \(1/\tau_{k_i}\) satisfy a linear system. Under (A.3) and \(\Delta(G_1) \neq 0\) of Theorem 1, \(1/\tau_{k_i} = 0\) and under (A.3) b) \(1/\tau_{k_{i+1}} = -1/\beta_{k_{i+1}}j_{k_{i+1}}\). The nonexistence of a solution for the system (1) translates into a zero value for some of the reciprocals.

If some coefficients in (1) are zero, a more careful procedure with subsets of equations is needed.

All null-hypotheses under which the inequality \(\Delta(6) \neq 0\) continues to hold are testable. But by imposition of additional over-identifying restrictions, especially covariance restrictions that make \(\Sigma\) into a diagonal matrix, we could lose identifiability.

For \(G=7\), Theorem 1 describes models with identifiable parameter either under 4-cyclical or 6-cyclical septuples. When \(G=8\) we also have cases identifiable under 8-cyclical or two disjoint 4-cyclical octuples.

If the structural model is the standard $B^{-1}x = u$, $u \sim i.i.d. (O, \Sigma)$, with 
$\mu_i = (\mu_i) = -B^{-1}\Gamma$, $\Omega_i = (\Omega_i) = B^{-1}\Sigma B^{-1}$, the results of this paper hold if there are $G-1$ restrictions on each row $(B_{\ell}, \Gamma_{\ell})$, $\ell = 1, \ldots, G$. If $n(\ell) + 1$ elements of $B_{\ell}$ are unknown, $n(\ell)$ elements in $\Gamma_{\ell}$ are zero, $0 \leq n(\ell) \leq G-2$, where $\beta_{\ell} = -1$. We need some new notation.

Let $(\beta_{\ell} = \beta_{\ell}m_1 \ldots \beta_{\ell}m_{n(\ell)})$ be the unknown elements in $B_{\ell}$ standing in the columns $(m_0, m_{1}, \ldots, m_{n(\ell)})$ and let $(\sigma_{\ell}m_1 \ldots \sigma_{\ell}m_{n(\ell)}) = 0$ be the zero elements in $\Gamma_{\ell}$ standing in the columns $(m_{1}, \ldots, m_{n(\ell)})$. The parameter $(TB, TP, T\Sigma T')$ is equivalent to $(B, \Gamma, \Sigma)$ under a cycle of covariance restrictions if and only if

$$T = I_{G} + PB^{-1}, \quad P_\ell = (0 \ldots \tau_0 \ldots 0 \ldots \tau_{n(\ell)} \ldots 0),$$

with the unknowns $(\tau_0, \tau_1, \ldots, \tau_{n(\ell)})$ in row $P_\ell$ standing in its columns $(m_0, m_{1}, \ldots, m_{n(\ell)})$ and satisfying

$$0 = (T\Gamma_{\ell})m_{ij} = (P\Pi_{\ell})m_{ij}, \quad j = 1, \ldots, n(\ell), \quad (4)$$

$$0 = (T\Sigma T')_{k_1k_{i+1}} = (PB^{-1}\Sigma)_{k_1k_{i+1}} + (PB^{-1}\Sigma)_{k_1k_{i+1}} + (P\Omega P')_{k_1k_{i+1}}, \quad i = 1, \ldots, G. \quad (5)$$

With $c_{\ell k}$, $k = 0, \ldots, n(\ell)$, the signed determinant of order $n(\ell)$ in

$$R_{\ell} \equiv \begin{pmatrix}
\pi_{j_0m_1} & \pi_{j_0m_2} & \cdots & \pi_{j_0m_{n(\ell)}} \\
\pi_{j_1m_1} & \pi_{j_1m_2} & \cdots & \pi_{j_1m_{n(\ell)}} \\
\cdots & \cdots & \cdots & \cdots \\
\pi_{j_{n(\ell)}m_1} & \pi_{j_{n(\ell)}m_2} & \cdots & \pi_{j_{n(\ell)}m_{n(\ell)}}
\end{pmatrix}$$

after deleting its row $(\Pi_{j_\ell m_1}, \cdots, \Pi_{j_\ell m_{n(\ell)}})$, the equations (4) imply
If there are no restrictions on $\Gamma_l$, put $c_{jn0}^{-1}$. Substituting into (5) we have $(TB, T\Gamma, T\Sigma T')$ is equivalent to $(B, \Gamma, \Sigma)$ under a $G$-cycle of covariance restrictions if and only if $\tau= (\tau_1, \tau_2, \ldots, \tau_G)$ satisfies the system

$$0 = \bar{a}_i \tau_{ki}^i + \bar{b}_i \tau_{ki+1}^i + \bar{d}_i \tau_{ki}^i \tau_{ki+1}^i, \quad i=1, \ldots, G,$$

(7)

where $\bar{a}_i \equiv (CB^{-1}\Sigma)_{ki}k_{i+1}$, $\bar{b}_i \equiv (CB^{-1}\Sigma)_{ki+1}k_i$, $\bar{d}_i \equiv (C\Omega C')_{ki}k_{i+1}$ and $C_G = (\ldots, 0, c_{j\ell 0}, 0, \ldots, c_{j\ell 1}, \ldots, 0, c_{j\ell n(l)}, 0, \ldots)$.

Again (7) is a system of $G$ bilinear equations in $G$ unknowns and if its solution $\tau=0$ is unique, the parameter $(B, \Gamma, \Sigma)$ is globally identifiable.

Special Cases

A. As in the proof of Theorem 1, the $G$ bilinear equations (7), with $G$ even, can be reduced to $G/2$ linear equations if

$$(CB^{-1}\Sigma)_{ki-1}k_{i} (C\Omega C')_{ki}k_{i+1} = (CB^{-1}\Sigma)_{ki+1}k_{i} (C\Omega C')_{ki-1}k_{i}, \text{ for } i \text{ even.}$$

(8)

This condition holds if $C_{ki-1} = C_{ki+1}$, i.e., if $j_{ki-1} \ell = j_{ki+1} \ell$, for $\ell = 0, 1, \ldots, n(k_{i-1}) \cap n(k_{i+1})$ and $R_{ki-1} = \Omega_{ki+1}$. This is the case if the rows $k_{i-1}$ and $k_{i+1}$ have unknown elements in the same columns of $B$ and zero elements in the same columns of $r$.

When $G=4$ there is only the condition that $C_{k1} = C_{k3}$. Any model in $F_a(S_1(4))$ can be modified into a model satisfying this condition. For example $[[3312] | S_1(4), \sigma_{34} \neq 0] \in F_a(S_1(4))$ can be modified into the model
in which \((j_{10}, j_{11}) = (j_{20}, j_{21}) = (3, 4), n(1) = n(2) = 1, \begin{pmatrix} m_{11} \\ m_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, R_1 = R_2 = \begin{pmatrix} \pi_{31} \\ \pi_{41} \end{pmatrix}\)

Its parameter is identifiable under local identifiability conditions (2)

\[
(\sigma_{31}\sigma_{44} - \sigma_{41}\sigma_{43})(-\sigma_{31}\sigma_{34} + \sigma_{41}\sigma_{33}) = 0.
\]

Other examples can be constructed from a member of \(F_a(S_1(4))\) by leaving its rows \(k_1\) and \(k_3\) unchanged and by replacing the exclusion restrictions on \(B_{k_2}\) and \(B_{k_4}\) by exclusion restrictions on \(\Gamma_{k_2}\) and \(\Gamma_{k_4}\).

B. If a model in \(F_b(S_1(4))\) satisfying \(j_{k_1} = k_3\) is modified by replacing the exclusion restrictions in \(B_4\) (or \(B_2\)) by exclusion restrictions in \(\Gamma_4\) (or \(\Gamma_2\)), the new model has an identifiable parameter. For example the model

\[
(B, \Gamma) = \begin{pmatrix} -1 & 0 & 0 & \sigma_{11} & \sigma_{12} \\ 0 & -1 & \beta_{23} & \beta_{24} & 0 & \sigma_{21} & \sigma_{22} \\ \beta_{31} & 0 & -1 & 0 & \sigma_{31} & 0 & 0 & \sigma_{33} & \sigma_{34} \\ \beta_{41} & \beta_{42} & \beta_{43} & -1 & 0 & 0 \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & \sigma_{34} \\ 0 & 0 & \sigma_{43} & \sigma_{44} \end{pmatrix}
\]

is a modification of the model \(\{[2312] \mid S_1(4), \beta_{23}\sigma_{34} 
\neq 0\} \in F_b(S_1(4))\). To see this when \(G = 4\), reduce the four equations (7) to two equations and then to the single equation

\[
\left( \prod_{l=1}^{4} \overline{a}_{l} - \prod_{l=1}^{4} b_{l} \right) \zeta_{k_1} = -\overline{a}_{1}\overline{a}_{2} (\overline{a}_3 d_4 - \overline{b}_4 d_3) - \overline{b}_3 \overline{b}_4 (\overline{a}_1 d_2 - \overline{b}_2 d_1) (\zeta_{k_1})^2
\]

With \(j_{k_1} = k_3, B_{k_3}(B^{-1}\Sigma)^{k_1} = 0, i = 2, 4, B_{k_3}(\Omega C)^{k_3} = \overline{b}_3\) imply \(\overline{a}_3 d_4 - \overline{b}_4 d_3 = -\overline{a}_3 \overline{b}_3, \overline{a}_1 d_2 - \overline{b}_2 d_1 = \overline{b}_2 \overline{a}_2\) and the coefficient of the quadratic term is
\[
\begin{align*}
\bar{a}_2 \bar{b}_3 ( -\bar{a}_1 \bar{a}_3 + \bar{b}_2 \bar{b}_4 ) &= \bar{a}_2 \bar{b}_3 \\
\begin{pmatrix}
(B^{-1} \Sigma)_{jk_1 k_2} & (B^{-1} \Sigma)_{jk_1 k_4} \\
(B^{-1} \Sigma)_{jk_3 k_2} & (B^{-1} \Sigma)_{jk_3 k_4}
\end{pmatrix} &= 0
\end{align*}
\]

and (10) has an identifiable parameter under the local identifiability conditions (2) i.e. when

\[
\begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{31} & \sigma_{32}
\end{pmatrix} \begin{pmatrix}
\sigma_{21} & \sigma_{22} \\
\sigma_{31} & \sigma_{32}
\end{pmatrix} \neq 0.
\]

Similar extensions of members in \( F_b(S_1(4)) \) stated above result in models with identifiable parameter. By special calculation, Mallela et al. (1993) established the identifiability of the model

\[
(B, \Gamma) = \begin{pmatrix}
-1 & 0 & 0 & \beta_{14} \sigma_{11} & \sigma_{12} \\
\beta_{21} & -1 & 0 & 0 & \sigma_{21} & \sigma_{22} \\
\beta_{31} & 0 & -1 & 0 & \sigma_{31} & \sigma_{32} \\
\beta_{41} & \beta_{42} & \beta_{43} & -1 & 0 & 0
\end{pmatrix}, \quad
\Sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & 0 & 0 \\
\sigma_{21} & \sigma_{22} & 0 & 0 \\
0 & 0 & \sigma_{33} & \sigma_{34} \\
0 & 0 & \sigma_{43} & \sigma_{44}
\end{pmatrix}
\]

This is an extension of the model \( \{41121|S_1(4), \beta_{14} \sigma_{43} \neq 0 \} \in F_b(S_1(4)) \) with \( j_{k_3} = k_1 \) and therefore it has an identifiable parameter under its local identifiability conditions (2) that are

\[
\beta_{14} (\sigma_{44} + \beta_{43} \sigma_{34}) (\sigma_{43} + \beta_{43} \sigma_{33}) (\begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{31} & \sigma_{32}
\end{pmatrix} + \beta_{14} \beta_{42} (\begin{pmatrix}
\sigma_{21} & \sigma_{22} \\
\sigma_{31} & \sigma_{32}
\end{pmatrix}) \neq 0.
\]

Unique identifiability under covariance restrictions is generally understood to be a property of recursive models or models where there is at least one equation identifiable without covariance restrictions. The parameters of the other equations can be further restricted by the zero correlations with residuals that are identifiable already. Without such first equation covariance restrictions in general imply local identifiability only and inequality restrictions must be relied upon for uniqueness.

Taking inspiration from the special model (11) analyzed by Mallela et al. (1993), we have characterized four families of models with parameter identifiable under a subset of uncorrelated residuals and without having to invoke inequality restrictions. In these families, not counting the zero correlation, there is one coefficient restriction missing per equation so that not one single equation is identifiable without the covariance restrictions. Further for all members of the family the string of zero correlations is a closed cycle of order at least four and the odd-numbered variables are siblings. For these families Theorem 1 states that the parameter is globally identifiable if it is locally identifiable.

We derived the complete membership list of four equation models with the above properties and we described by example how models with five and six equations are to be recognized as members of the above families. Since the analysis can be applied immediately to models that contain external variables, the results of this paper are important in modeling price and quantity variables under first and second moment reproducing inverse demand and supply equations with intercorrelated inverse demand shocks and intercorrelated supply shocks that are uncorrelated with the demand shocks. Models (9) and (10) provide an illustration.
Appendix.

1. Total Number of $G_1$-cyclical $G$-tuples in $G \times G \Sigma$.

A. The number of $G_1$-cycles is $G!/2G_1(G-G_1)!$.

To see this, a $G_1$-cycle in $G \times G \Sigma$ has two elements in a row or column. The $G_1$-cycle with vertices $(\sigma_{11}, \sigma_{1j})$ contains the vertices

$$\sigma_{1i} = \sigma_{i3} = \ldots = \sigma_{m_{k-1}j} = \sigma_{j1}, \quad m_1 = m_{k+1} = 1, \ m_2 = i, \ m_k = j.$$ 

Given the subscripts $1, i, j$, the remaining subscripts are $G-3$ integers out of $G-3$ possible, which implies the pair $(\sigma_{1i}, \sigma_{1j}), 1 < i < j \leq G$, can be the side of $(G-3)(G-4)\ldots [G-(G-1)]$ different $G_1$-cycles. When $i=2$ there are $G-2$ different values for $j$, when $i=3$ there are $G-3$ different values for $j$ and for $i=G-1$, $j=G$ is the only value of $j$. Therefore the first row pairs $(\sigma_{1i}, \sigma_{1j}), 1 < i < j$, are a side of

$$(G-3)(G-4)\ldots [G-(G-1)]\ldots [G-(G-1)-(G-1)+1] = \frac{1}{2} (G-1)(G-2)(G-3)\ldots [G-(G-1)]$$

different $G_1$-cycles. When $G=3$, this is $(G-1)$#.

Next, the pair $(\sigma_{2i}, \sigma_{2j}), 2 < i < j, j \leq G$, can be combined with $G-3$ integers out of $G-4$ to form $(G-4)(G-5)\ldots (G-G_1)$ different polygons, also different from the polygons constructed before. When $i=3$ there are $G-3$ different values for $j$ and when $i=G-1$ there is only one value for $j$. Therefore the second row pairs $(\sigma_{2i}, \sigma_{2j}), 2 < i < j$, are a side of

$$(G-4)(G-5)\ldots (G-G_1)\ldots [G-(G-1)-(G-1)+1] = \frac{1}{2} (G-2)(G-3)(G-4)\ldots (G-G_1)$$

different $G_1$-cycles.

In this sequence the last row pairs are $(\sigma_{G-(G-1)i}, \sigma_{G-(G-1)j})$, $G-(G-1) < i < j, j \leq G$, which are a side of $\frac{1}{2} (G-1)(G-2)\ldots (1)$ different cycles. Therefore, adding over the sequence, the number of different $G_1$-cycles in $G \times G \Sigma$ is

$$\frac{1}{2} \sum_{m=G_1}^{G} (m-1)(m-2)\ldots [m-(G-1)] = \frac{1}{2G_1} G(G-1)(G-2)\ldots [G-(G-1)].$$
B. The number of connected G-tuples per $G_1$-cycle is $G_1 G^{G- (G_1 + 1)}$.

Each polygon of order $G_1 < G$ can be combined with $G^G - G^{G_1}$ different choices of G-k restrictions to constitute assignable and not assignable sets of G restrictions.

Clearly the number of $G_1$-laterals per polygon of order $G_1$ is one. Renumbering the subscripts so that the vertices of the polygon stand in the first $G_1$ rows and columns, let the polygon of order $G_1$ be denoted

$$p_{G_1} = \sigma_{G_1} \cdot \sigma_{G_1 + 1} \cdot \ldots = \sigma_{G_1} .$$

Connected G-tuples linked to the $G_1$-cycle number

i) $G_1$ in the form

$$p_{G_1} = \sigma_{G_1 + 1} \cdot n_{G_1 + 1} = 0, \quad n_{G_1 + 1} \in \{1, 2, \ldots, G_1\}, \quad \text{when } G = G_1 + 1,$$

ii) $G_1 (G_1 + 1)$ and $G_1$ respectively in the form

$$p_{G_1} = \sigma_{G_1} \cdot n_{G_1 + 1} = \sigma_{G_1 + 2} \cdot n_{G_1 + 2} = 0, \quad n_{G_1 + 2} \in \{1, 2, \ldots, G_1, G_1 + 1\}, \quad \text{or}$$

$$p_{G_1} = \sigma_{G_1 + 2} \cdot n_{G_1 + 1} = \sigma_{G_1 + 1} \cdot n_{G_1 + 2} = 0, \quad \text{when } G = G_1 + 2,$$

iii) $G_1 (G_1 + 1) (G_1 + 2)$, $G_1 (G_1 + 1)$, $G_1 (G_1 + 2)$, $G_1 (G_1 + 1)$, $G_1$, $G_1$, $G_1$ respectively in the form

$$p_{G_1} = \sigma_{G_1} \cdot n_{G_1 + 1} = \sigma_{G_1 + 2} \cdot n_{G_1 + 2} = \sigma_{G_1 + 3} \cdot n_{G_1 + 3} = 0, \quad n_{G_1 + 3} \in \{1, 2, \ldots, G_1, G_1 + 1, G_1 + 2\},$$

or

$$p_{G_1} = \sigma_{G_1} \cdot n_{G_1 + 1} = \sigma_{G_1 + 3} \cdot n_{G_1 + 2} = \sigma_{G_1 + 2} \cdot G_1 + 3 = 0, \quad \text{or}$$

$$p_{G_1} = \sigma_{G_1 + 2} \cdot n_{G_1 + 1} = \sigma_{G_1 + 1} \cdot G_1 + 2 = \sigma_{G_1 + 3} \cdot n_{G_1 + 3} = 0, \quad \text{or}$$
\[ p_{G_1} = \sigma_{G_1+2} n_{G_1+1} = \sigma_{G_1+3} n_{G_1+2} = \sigma_{G_1+1} G_1+3 = 0, \text{ or} \]
\[ p_{G_1} = \sigma_{G_1+3} n_{G_1+1} = \sigma_{G_1+1} G_1+3 = \sigma_{G_1+2} G_1+1 = 0, \text{ or} \]
\[ p_{G_1} = \sigma_{G_1+3} n_{G_1+1} = \sigma_{G_1+1} G_1+3 = \sigma_{G_1+2} G_1+3 = 0, \text{ or} \]
\[ p_{G_1} = \sigma_{G_1+3} n_{G_1+1} = \sigma_{G_1+2} G_1+3 = \sigma_{G_1+1} G_1+2 = 0, \text{ for } G = G_1 + 3. \]

Adding all forms shows the result.

2. Proof of Lemma 3.

We proceed in steps. With a single covariance restriction \((T \Sigma T')_{k_1 k_2} = 0\) we have \(\tau_{k_1} = 0, \tau_{k_2} = 0\) only if \((j_{k_1}, k_2) = 0\). Also \(\tau_{k_1} = 0, \tau_{k_2} = 0\) only if \((j_{k_2}, k_1) = 0\), or the parent of the \(k_2\)th variable is exogenous in the \(k_1\)th equation. Therefore, we have

\[ (T \Sigma T')_{k_1 k_2} = 0, \ (j_{k_1}, k_2) \neq 0, \Rightarrow \quad \text{not} \ (\tau_{k_1} = 0, \tau_{k_2} = 0) \]
\[ (T \Sigma T')_{k_1 k_2} = 0, \ (j_{k_2}, k_1) \neq 0, \Rightarrow \quad \text{not} \ (\tau_{k_1} = 0, \tau_{k_2} = 0) \]

and the combined conclusion that:

\[ (j_{k_1}, k_2)(j_{k_2}, k_1) \neq 0, \ (T \Sigma T')_{k_1 k_2} = 0 \quad \text{and} \quad (\tau_{k_1}, \tau_{k_2}) \neq 0 \quad \Rightarrow \quad \tau_{k_1}, \tau_{k_2} \neq 0. \ (2.1) \]

If two connected covariance restrictions \(\sigma_{k_1 k_2} = \sigma_{k_2 k_3} = 0\) are imposed, the two equations \((T \Sigma T')_{k_1 k_2} = (T \Sigma T')_{k_2 k_3} = 0\) are

\[ 0 = (j_{k_1}, k_2) \tau_{k_1} + (j_{k_2}, k_1) \tau_{k_2} + \tau_{k_1} \tau_{k_2} \omega_{j_{k_1} j_{k_2}} \]
\[ 0 = (j_{k_3}, k_2) \tau_{k_3} + (j_{k_2}, k_3) \tau_{k_2} + \tau_{k_3} \tau_{k_2} \omega_{j_{k_3} j_{k_2}} \]

By the argument applied to \((T \Sigma T')_{k_1 k_2} = 0\) leading to \((2.1)\), conclude

\[ (j_{k_1}, k_2)(j_{k_2}, k_3)(j_{k_3}, k_2)(j_{k_2}, k_1) \neq 0, \quad (T \Sigma T')_{k_1 k_2} = (T \Sigma T')_{k_2 k_3} = 0 \quad \text{and} \quad (\tau_{k_1}, \tau_{k_2}, \tau_{k_3}) \neq 0 \quad \Rightarrow \quad \tau_{k_1} \tau_{k_2} \tau_{k_3} \neq 0. \]
With three cyclical covariance restrictions \( \sigma_{k_1k_2} = \sigma_{k_2k_3} = \sigma_{k_3k_1} = 0 \) we have the implications

\[
\begin{align*}
(j_{k_1}, k_2) \neq 0 & \Rightarrow \text{ "not } (\tau_{k_1} \neq 0, \tau_{k_2} = 0)\text{"}, & (j_{k_2}, k_1) \neq 0 & \Rightarrow \text{ "not } (\tau_{k_1} = 0, \tau_{k_2} \neq 0)\text{"}, \\
(j_{k_2}, k_3) \neq 0 & \Rightarrow \text{ "not } (\tau_{k_2} \neq 0, \tau_{k_3} = 0)\text{"}, & (j_{k_3}, k_2) \neq 0 & \Rightarrow \text{ "not } (\tau_{k_2} = 0, \tau_{k_3} \neq 0)\text{"}, \\
(j_{k_3}, k_1) \neq 0 & \Rightarrow \text{ "not } (\tau_{k_3} \neq 0, \tau_{k_1} = 0)\text{"}, & (j_{k_1}, k_3) \neq 0 & \Rightarrow \text{ "not } (\tau_{k_3} = 0, \tau_{k_1} \neq 0)\text{"}.
\end{align*}
\]

Therefore,

\[
(T \Sigma T')_{k_1 k_2} = (T \Sigma T')_{k_2 k_3} = (T \Sigma T')_{k_3 k_1} = 0, 
\text{ and either}
\]

\[
(j_{k_1}, k_2)(j_{k_2}, k_3)(j_{k_3}, k_1) \neq 0 \text{ or } (j_{k_2}, k_1)(j_{k_3}, k_2)(j_{k_1}, k_3) \neq 0 \Rightarrow \tau_{k_1} \tau_{k_2} \tau_{k_3} \neq 0.
\]

To see this, if \( (\tau_{k_1}, \tau_{k_2}, \tau_{k_3}) \neq 0 \) with \( \tau_{k_1} = 0 \), then \( (j_{k_1}, k_2) \neq 0 \) implies \( \tau_{k_2} \neq 0 \) and then \( (j_{k_2}, k_3) \neq 0 \) implies \( \tau_{k_3} \neq 0 \). If \( (\tau_{k_1}, \tau_{k_2}, \tau_{k_3}) \neq 0 \) with \( \tau_{k_2} = 0 \), then \( (j_{k_2}, k_3) \neq 0 \) implies \( \tau_{k_3} \neq 0 \) and then \( (j_{k_3}, k_1) \neq 0 \) implies \( \tau_{k_1} \neq 0 \). Finally, if \( (\tau_{k_1}, \tau_{k_2}, \tau_{k_3}) \neq 0 \) because \( \tau_{k_3} \neq 0 \), then \( (j_{k_3}, k_1) \neq 0 \) implies \( \tau_{k_1} \neq 0 \) and then \( (j_{k_1}, k_2) \neq 0 \) implies \( \tau_{k_2} \neq 0 \). We could also use the implications from the second column above to prove the or part of the statement.

The same argument goes through for general \( G_1 \)-cycles.

3. Relations between elements of \( \mathfrak{E}_3, \Sigma, B^{-1} \Sigma, \text{ and } \Omega \).

The equations (1) to be solved contain elements of \( B^{-1} \Sigma \) and of \( \Omega \). Observe the following properties:

(P1) \( B_m \Omega^j = (j, m) = -\omega_{mj} + \beta_{mj} \omega_{mj} \).

(P2) \( a = B_m (B^{-1} \Sigma)^n = -(m, n) + \beta_{mj} (j, m) \)

\[
\sigma_{nm} = B_n (B^{-1} \Sigma)^m = -(n, m) + \beta_{nj} (j, m)
\]

\( \sigma_{mn} = 0 \) implies \( (m, n) \omega_{mj} = (j, m) \omega_{mj} = (j, m) (j, m) \).

(P3) \( \sigma_{mn} = 0, (m, n) \neq 0, \beta_{mj} \neq 0 \) imply

\[
(j, m) (-1/\beta_{mj}) + (j, m) (-\omega_{mj} / (m, n)) = -\omega_{mj}.
\]
4. Proof of Theorem 1.

We have to show that the solution \( \tau(G_1) = (\tau_{k_1}, \tau_{k_2}, \ldots, \tau_{k_{G_1}}) = 0 \) of the system \((T \Sigma T)'_{k_1 k_{i+1}} = 0, \ i = 1, \ldots, G_1\) is unique under the stated conditions. This is the system of \( G_1 \) equations

\[
 a_i \, \tau_{k_i} + b_i \, \tau_{k_{i+1}} = -\tau_{k_1} \, \tau_{k_{i+1}} \omega_{j_i j_{i+1}}, \quad i = 1, \ldots, G_1. \tag{4.1}
\]

Observe that the Jacobian matrix of these equations at \( \tau(G_1) = 0 \) has determinant \( \Delta(G_1) = z_1(G_1) - z_2(G_1) \). By assumption, either \( a_i \neq 0, \ i = 1, \ldots, G_1 \), or \( b_i \neq 0, \ i = 1, \ldots, G_1 \). Therefore from Lemma 1, \( \tau(G_1) = 0 \) implies each component of \( \tau(G_1) \) is different from zero.

Eliminating \( \tau_{k_1} \neq 0 \) from the equations corresponding to \((T \Sigma T)'_{k_{i-1} k_1} = (T \Sigma T)'_{k_1 k_{i+1}} = 0\), we have the \( G_1/2 \) equations

\[
\begin{bmatrix}
 a_{i-1} & \omega_{j_{k_{i-1}} j_{k_i}} \\
 -b_{i} & \omega_{j_{k_{i-1}} j_{k_i}} \\
 j_{k_{i-1}} & j_{k_i}
\end{bmatrix}
\tau_{k_{i-1}} \tau_{k_{i+1}}
\]

\[ + a_{i-1} a_i \, \tau_{k_{i-1}} - b_{i-1} b_i \, \tau_{k_{i+1}} = 0, \quad i = 2, 4, \ldots, G_1. \tag{4.2, i} \]

From (P2), Appendix 3. we have
Hence, when \( \sigma_{k_n k_1} = 0 \) adding (4.3) to (4.2,i), the latter is equivalent to

\[
\{ [a_{i-1} \omega_{j_{k_i} j_{k_{i+1}}} - (j_{k_n} k_i) \omega_{k_n j_{k_i}}] - [b_i \omega_{j_{k_{i-1}} j_{k_i}} - (k_n k_i) \omega_{j_{k_i} j_{k_n}} ]
\]

\[-(j_{k_i} k_n) (j_{k_n} k_i)] \tau_{k_{i-1}} \tau_{k_{i+1}} + a_{i-1} a_i \tau_{k_{i-1}} \tau_{k_{i+1}} + b_{i-1} b_i \tau_{k_{i+1}} = 0. \tag{4.4,i,n}
\]

or subtracting (4.3) from (4.2,i), the latter is also equivalent to

\[
\{ [a_{i-1} \omega_{j_{k_i} j_{k_{i+1}}} - (k_n k_i) \omega_{j_{k_i} j_{k_n}} ] - [b_i \omega_{j_{k_{i-1}} j_{k_i}} - (j_{k_n} k_i) \omega_{j_{k_i} j_{k_n}} ]
\]

\[+ (j_{k_i} k_n) (j_{k_n} k_i)] \tau_{k_{i-1}} \tau_{k_{i+1}} + a_{i-1} a_i \tau_{k_{i-1}} \tau_{k_{i+1}} - b_{i-1} b_i \tau_{k_{i+1}} = 0. \tag{4.5,i,n}
\]

The first quadratic term does not depend on elements of \( \Omega \) if

1. \( k_{i-1} \) and \( k_{i+1} \) are siblings i.e.

\[ j_{k_{i-1}} = j_{k_{i+1}} \quad \text{and then from (4.2,i)} \]

\[ a_{i-1} a_i \tau_{k_{i-1}} - b_{i-1} b_i \tau_{k_{i+1}} = 0, \tag{4.6,i} \]

2. \( k_{i-1} \) or a sibling of \( k_{i-1} \) (for \( n \neq i-1 \)) is the parent of \( k_{i+1} \):

\[ j_{k_{i-1}} = j_{k_n}, k_n = j_{k_{i+1}} \quad \text{and then from (4.4,i,n) provided } \sigma k_i k_n = 0, \]

\[ a_{i-1} a_i \tau_{k_{i-1}} - b_{i-1} b_i \tau_{k_{i+1}} = c_{i n} \tau_{k_{i-1}} \tau_{k_{i+1}}, \quad c_{i n} = a_{i-1} (j_{k_i} k_n), \tag{4.7,i,n} \]

3. \( k_{i+1} \) or a sibling of \( k_{i+1} \) (for \( n \neq i+1 \)) is the parent of \( k_{i-1} \):

\[ j_{k_{i+1}} = j_{k_n}, k_n = j_{k_{i-1}} \quad \text{and then from (4.5,i,n) provided } \sigma k_i k_n = 0, \]
The models with possible unique solution $\tau(G_1)=0$ must be those for which the bilinear system consisting of $G_1/2$ equations and unknowns from the three equations above can be reduced further to a linear system of fewer equations and unknowns. These models are:

a) $[m,j_k,k_2,m,j_k,k_4,...,m,j_k,k_{G_1}]$ which has $j_{k_{i-1}}=j_{k_{i+1}}=m$, with the parent $m$ any one of the variables $(k_2,k_4,...,k_{G_1})$. The coefficient of $\tau k_{i-1} \tau k_{i+1}$ is zero and (4.6,i)) is a linear equation in $\tau k_{i-1}$ and $\tau k_{i+1}$. When the variables $(k_1,k_3,...,k_{G_1-1})$ are all siblings, the equations form a linear homogeneous system in $\tau k_1, \tau k_3,...,\tau k_{G_1-1}$, with matrix having $z_1(G_1)-z_2(G_1)$ as determinant. The solution $\tau(G_1)=0$ is unique if and only if this is different from zero.

b) The adjacent equations (4.8,i,n) and (4.7,i+2,n) correspond to the model assumption that $k_{i+1}$ and $k_n$ are siblings with $k_n$ the parent of $k_{i-1}$ and $k_{i+3}$, not excluding the possibility that $k_n=k_{i+1}$, and provided $\sigma k_{i} k_{n}=0$. Eliminating $t_{k_{i+1}}$ from these two equations, we get the equation

$$a_{i-1} a_{i+1} a_{i+2} \tau k_{i-1} - b_{i-1} b_{i+1} b_{i+2} \tau k_{i-1} = d_1 \tau k_{i-1} \tau k_{i+3}, \quad (4.9,i)$$

with

$$d_1 = a_{i-1} a_{i} c_{i+2,n} - b_{i+1} b_{i+2} d_{in} = a_{i-1} a_{i} a_{i+1} (j_{k_{i+2}},k_n) - b_{i+1} b_{i+2} b_{i} (j_{k_{i}},k_n),$$

$$= (j_{k_{i+1}},k_i) (j_{k_{i+1}},k_{i+1}) (j_{k_{i+1}},k_{i+2}) (j_{k_{i+2}},k_n) - (j_{k_{i+2}},k_{i+1}) (j_{k_{i+3}},k_{i+2}) (j_{k_{i+1}},k_i) (j_{k_{i}},k_n)$$

$$- (j_{k_{i+2}},k_{i+1}) (j_{k_{i+3}},k_{i+2}) (j_{k_{i+1}},k_i) (j_{k_{i}},k_n)$$
\[
= (j_{k_1}, k_{i+1})(j_{k_1+2}, k_n) \begin{vmatrix}
(j_{k_1-1}, k_i) (j_{k_1+3}, k_{i+2}) \\
(j_{k_1+1}, k_i) (j_{k_1+2}, k_i+2)
\end{vmatrix}, \text{ if } k_n = k_{i+1} \text{ or } j_{k_1} = j_{k_1+2}.
\]

\[
= (j_{k_1}, k_{i+1})(j_{k_1+2}, k_n) \begin{vmatrix}
(k_n, k_i) (k_n, k_{i+2}) \\
(j_{k_n}, k_i) (j_{k_n}, k_{i+2})
\end{vmatrix} = 0.
\]

from (P4) of Appendix 3. provided \(\sigma_{k_n k_{i+2}} = 0\).

When \(k_n = k_{i+1}\), both covariances are zero and the equation (4.9,i) is linear. For instance, when \(i=2, n-3\) and we have the model \([k_3, j_k_2, j_k_3, j_k_4, k_3, j_k_6, \ldots, k_3, j_k_{G_1}]\) with the vector of unknowns \((\tau_{k_1}, \tau_{k_5}, \tau_{k_7}, \ldots, \tau_{k_{G_1}})\) satisfying the homogeneous linear system of equations (4.9,2) and (4.6,i), \(i=6, \ldots, G_1\). Again \(z_1(G_1) - z_2(G_1) \neq 0\) implies the null solution is unique.

c) When \(j_{k_1} = j_{k_1+2}\) the equation (4.9,i) is linear if \(\sigma_{k_n k_{i+2}} = 0\).

For example if \(i=2, n=6\) we have the model \([k_6, j_k_2, j_k_6, j_k_6, j_k_6, j_k_6, j_k_{G_1}]\).

d) The adjacent equations (4.7,i,n) and (4.8,i+2,n) correspond to the model assumption that \(k_{i-1}, k_n\) and \(k_{i+3}\) are siblings with \(k_n\) the parent of \(k_{i+1}\), not excluding the possibility that \(k_n\) is either \(k_{i-1}\) or \(k_{i+3}\), and provided \(\sigma_{k_i k_n} = \sigma_{k_{i+2} k_n} = 0\) in the population. Eliminating \(\tau_{k_{i+1}}\) from these two equations, we get the equation

\[a_{i-1} a_i a_{i+1} a_{i+2} \tau_{k_{i-1}} - b_{i-1} b_i b_{i+1} b_{i+2} \tau_{k_{i+3}} = -d_3 \tau_{k_{i-1}} \tau_{k_{i+3}}, \quad (4.10,i)\]

with

\[
d_3 = a_{i-1} a_i d_{i+2,n} - c_{i,n} b_{i+1} b_{i+2} = a_{i-1} b_{i+2} \begin{vmatrix}
(j_{k_1}, k_{i+1}) (j_{k_1}, k_n) \\
(j_{k_1+2}, k_{i+1}) (j_{k_1+2}, k_n)
\end{vmatrix}.
\]
The equation (4.10, i) is linear if \( j_{k_1} = j_{k_{i+2}} \), or \( k_i = j_{k_{i+2}} \), or \( k_{i+2} = l_k \) with \( \sigma_{k_1 k_{i+2} k_{i+4}} \sigma_{k_{i+2} k_{i+4}} = 0 \). Corresponding to the given model, the equations (4.10, 2) and (4.6, i), \( i = 6, 8, ..., G_1 \), determine values of \( (\tau_{k_1}, \tau_{k_2}, ..., \tau_{k_{G_1-1}}) \). If \( z_1(G_1) - z_2(G_1) \neq 0 \), the zero solution is unique.
References:


