ASYMPTOTIC VALUE OF MIXED GAMES*

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In this paper we are concerned with mixed games, i.e., games with on one hand an "ocean" of insignificant players (formalized by a continuum of players) and on the other hand some significant players (atoms). Considering these games as limits of finite games, we show, for the subset pFL, that the Shapley-Hart value of the mixed game corresponding to the uniform probability measure is the limit of the Shapley values of the associated finite games. This paper should then be considered as a generalization of the results of the work by Aumann-Shapley on nonatomic games.

1. Introduction. In 1960, Shapiro and Shapley proved a limit theorem on weighted majority games (1978), namely that if the players are separated into two categories: $N$, the major players on the one hand (who are supposed to be in finite number), and $M$, the minor players on the other hand, and if the number of minor players increases (their weights decreasing in an appropriate fashion) then the value of a major player in the (finite) game goes to a fixed limit (which does not depend on the way the weights of minor players go to zero).

In 1961 Milnor and Shapley (1978) introduced for majority games a notion of infinite game: the so-called "oceanic" games where the players are partitioned in "major players" (in finite number) and an "ocean" of infinitesimal minor players. They then used the majority rule to define a "value" for this game by generalizing the Shapley value of finite majority games.

The notion of value for infinite games was next precisely formalized and studied in Aumann-Shapley's book (1974). But, there, they are mainly concerned with nonatomic games, i.e., games where all players are infinitesimal.

From these contributions (see also a paper by Kannai (1966)) it appears clear that the notion of value for infinite games can be tackled in two different ways:

— the axiomatic approach which would define a value by a set of axioms.

— the asymptotic approach which consists in considering the infinite game as the limit of finite games and then calculating its value as the limit of their Shapley values.

It has been proved by Aumann-Shapley (1974) that on a particular set of nonatomic games these two approaches coincide.

Now, for mixed games, i.e., games where the set of players is made of major players (atoms) on the one hand and minor players (ocean) on the other, the axiomatic study was made by Hart (1973). He proved that for these games it is no longer possible to speak of the value of a game: there is an infinity of values, one for each probabilistic way of "mixing" the atoms into the infinitesimal players.

However, the Milnor-Shapley study (1978) in the case of majority games suggested that the asymptotic approach would lead to selecting the value associated with the uniform probability measure.

In this paper, we use the formalism of Hart (1973) and Aumann-Shapley (1974) to make the asymptotic study and we prove that on a particular subset of mixed games...
the two approaches coincide, in the case where the value associated to the uniform probability measure is used.

In §2, some definitions and notation are given. In §3, the main theorem is stated and proved.

II. Preliminaries. All definitions and notation are as in Aumann-Shapley (1974) and Hart (1973).

Let \((I, \mathcal{C})\) be a measurable space assumed to be isomorphic to \(([0, 1], \mathcal{B})\) where \(\mathcal{B}\) is the \(\sigma\)-field of Borel sets on \([0, 1]\): elements of \(I\) are players, those of \(\mathcal{C}\) are coalitions.

A set function or game is a real valued function \(v\) on \(\mathcal{C}\) such that \(v(\emptyset) = 0\).

A game \(v\) is monotonic if: \(\forall S, T \in \mathcal{C}, S \subseteq T \Rightarrow v(S) < v(T)\) (if \(Q\) is a space of games, \(Q^+\) will denote the space of monotonic games in \(Q\)).

A game \(v\) is of bounded variation if it is the difference between two monotonic games. The space of all games of bounded variation is called \(BV\).

On \(BV\) we define a norm, which we call the variation norm by: \(\|v\| = \inf\{u(I) + w(I) / u, w \in BV^+, v = u - w\}\), for all \(v\) in \(BV\). In the following we will always be concerned with the topology induced by the variation norm.

The space of all real valued functions \(f\) of bounded variation on \([0, 1]\) such that: \(f(0) = 0\) is called \(bv\). \(bv'\) will denote the space of those functions in \(bv\) which are continuous at 0 and 1.

A carrier of a game \(v\) is a coalition \(N\) such that: \(\forall S \in \mathcal{C}, v(S) = v(S \cap N)\).

The subspace of \(BV\) consisting of all measures with a finite carrier will be denoted \(FC\).

A coalition \(S\) is null if its complement is a carrier and a player \(s\) is null if \(\{s\}\) is null.

A game is nonatomic if any player is null.

We shall denote by \(NA\) the subspace of \(BV\) consisting of all nonatomic measures and \(FL = NA + FC\) (i.e., \(FL\) is the subspace of all measures that can be written as the sum of two measures, one nonatomic and the other with a finite carrier).

We call:
- \(bv'NA\) (resp. \(bv'FL\)) the closed subspace of \(BV\) spanned\(^1\) by games of the form \(f \circ \omega\), where \(f \in bv'\) and \(\omega \in NA^+\) (resp. \(\omega \in FL^+\)) is a probability measure.

Games in \(bv'FL\) are called mixed games.

A mixed game is thus any game of the form \(f \circ \omega\)—or linear combination or limit of such games—where \(\omega = \mu + \nu, \mu \in NA^+, \nu \in FC^+\). The finite carrier of \(v\) will represent the major or significant players (or atoms). The measure \(\mu\) will represent the "weights" of the minor players or "ocean."

- \(pNA\) (resp. \(pFL\)) the closed subspace of \(bv'NA\) (resp. \(bv'FL\)) spanned by all powers of measures in \(NA^+\) (resp. \(FL^+\)).

We shall need the following result of Hart (1973):

**Theorem A.** Let \(p\) be a continuous probability measure\(^2\) on \(([0, 1], \mathcal{B})\). Then there exists a value \(\Phi\) on \(bv'FL\) such that: \(\forall f \in bv', \forall \omega \in FL^+, \Phi v = \Phi_f v\) where \(v = f \circ \omega\) (i.e., there exists a value \(\Phi\) on \(bv'FL\) whose restriction to games of the form \(f \circ \omega\) is \(\Phi_f\)) where \(\Phi_f\) is defined in the following way:

let \(\mu \in NA^+\) and \(v \in FC^+\) be such that: \(\omega = \mu + \nu\) and denote by \(N\) the (finite) carrier of \(v\) (in its cardinality);

let \(\Pi\) be the set of all one-to-one mappings from \(N\) onto \(J_n = \{1, \ldots, n\}\);

let \(T_1, \ldots, T_n\) be \(n\) independent, equally distributed random variables, with distribu-

\(^1\) The space spanned by \(A \subseteq BV\) is the closure (in the variation norm topology) of the vector space generated by \(A\).

\(^2\) Its distribution function is continuous.
tion $p$, and $T^{(1)}, \ldots, T^{(n)}$ ($0 < T^{(1)} \leq \ldots \leq T^{(n)} < 1$) the corresponding order statistics.

Then the set function $\Phi_p v$ is defined by:

\[
\begin{align*}
\Phi_p v((s)) &= E \left[ \frac{1}{n!} \sum_{\pi \in \Pi} \Delta(\pi s, T^{(\pi)} v \circ \pi^{-1} f) \right] \quad \forall s \in N, \\
\Phi_p v(S) &= \alpha \mu(S) \quad \forall S \subset I/N,
\end{align*}
\]

with

\[
\alpha = \begin{cases} 
\frac{f(1) - \Phi_p v(N)}{\mu(I)} & \text{if } \mu(I) > 0, \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\Delta(i, t, \xi, f) = f \left[ t f(1 - \xi(J_n)) + \xi(J_i) \right] - f \left[ t f(1 - \xi(J_n)) + \xi(J_{i-1}) \right]
\]

(where $J_k = \{1, \ldots, k\}$).

We will now introduce the notion of asymptotic value (see Aumann-Shapley (1974)—chapter III). A partition $\pi$ of $I$ is a finite family of disjoint subsets whose union is $I$. A partition $\pi_2$ is a refinement of a partition $\pi_1$ if each member of $\pi_1$ is a union of members of $\pi_2$. A partition is measurable if each of its members is measurable. A sequence $(\Pi_1, \Pi_2, \ldots)$ of partitions is decreasing if $\Pi_{i+1}$ is a refinement of $\Pi_i$ for all $i$. This sequence is separating if $\forall s, t \in I, s \neq t, \exists i: s$ and $t$ are in different members of $\Pi_i$. A decreasing separating sequence of measurable partitions will be called an admissible sequence. Let $v$ be a game and $\Pi$ a measurable partition of $I$. We denote $v_{\Pi}$ the finite game whose players are the members $A_1, \ldots, A_m$ of $\Pi$, given by:

\[
\forall \Xi \subset \Pi: v_{\Pi}(\Xi) = v \left( \bigcup_{A_j \in \Xi} A_j \right).
\]

If $u$ is a finite game, $\varphi u$ will denote its Shapley value (1953).

Let $T$ be a measurable subset in $I$ and $\mathcal{\mathcal{P}} = (\Pi_1, \Pi_2, \ldots)$ a decreasing sequence of measurable partitions such that:

\[
\Pi_1 = (T, I \setminus T),
\]

\[
\forall m: T_m = \{ \Xi \in \Pi_m : \Xi \subset T \}.
\]

$T_m$ is a coalition in the finite game $v_{\Pi_m}$ and thus has a value $\varphi v_{\Pi_m}(T_m)$; if this value has a limit when $m$ goes to infinity one will denote:

\[
\varphi v(T) = \lim_{m \to +\infty} \varphi v_{\Pi_m}(T_m).
\]

If $\varphi v(T)$ exists for all admissible $\mathcal{\mathcal{P}}$ starting from $\Pi_1$ and is independent of $\mathcal{\mathcal{P}}$ then we call it the asymptotic value $\varphi v(T)$.

If all $T \in \mathcal{\mathcal{P}}$ possesses an asymptotic value then $v$ is said to have an asymptotic value. We will denote: ASYMP the set of all $v \in BV$ having an asymptotic value.

III. The main theorem.

**Theorem 1.** $pFL \subset ASYMP$ and the value on $pFL$ associated with the uniform probability on $[0, 1]$ coincides with the asymptotic value.

**Proof.** $pFL$ is spanned by the games of the form $f \circ \omega$ where $f$ is a polynomial function in $b_{v'}$ and $\omega$ belongs to $FL^*$. The following two lemmas proved in Aumann-Shapley (1974) enable us to limit ourselves to study games of this form only.
**Lemma 1.** \( ||\varphi|| < ||v|| \forall v \in \text{ASYMP}.\)

**Lemma 2.** \( \text{ASYMP} \) is a closed linear subspace of \( BV. \)

Hence, for proving our theorem, it is sufficient to show:

- \( \forall f \) polynomial function in \( bv' \)
- \( \forall \omega \) probability measure in \( FL^+ \):
  - (i) \( f \circ \omega \in \text{ASYMP} \).
  - (ii) \( \varphi(f \circ \omega) = \Phi(f \circ \omega) \),

where \( \Phi \) denotes the value \( \Phi_p \) associated with the uniform probability measure on \([0, 1]\).

Let us quote now a classical property of order statistics that allows us to give the precise expression of \( \Phi \).

**Lemma 3.** Let \( T_0 \) be a random variable whose distribution is uniform on \([0, 1]\). Let \((T_1, \ldots, T_n)\) be a random sample of size \( n \) from \( T_0 \) and \((T^{(1)}, \ldots, T^{(n)})\) be the associated order statistics. Then:

- The joint density of \((T^{(1)}, \ldots, T^{(n)})\) is given by:
  \[
g(t_1, \ldots, t_n) = \begin{cases} n! & \text{if } 0 < t_1 < \ldots < t_n < 1, \\ 0 & \text{otherwise.} \end{cases}
\]

- The marginal density of \( T^{(i)} \) is given by:
  \[
g_i(t) = \begin{cases} n! \frac{t^{i-1}(1-t)^{n-i}}{(i-1)!(n-i)!} & \text{on } [0, 1]; \\ 0 & \text{otherwise.} \end{cases}
\]

Applying this lemma to \( V = f \circ \omega, \omega = \mu + \nu, \mu \in NA^+, \nu \in FL^+ \) we have:

- \( \Phi_V(\{s\}) = \sum_{\pi \in \Pi} \int_0^1 \Delta(\pi s, t, \nu \circ \pi^{-1}, f) \frac{t^{\pi s - 1}(1-t)^{n-\pi s}}{(\pi s - 1)!(n-\pi s)!} \, dt \quad \forall s \in N, \) (1)

- \( \Phi_V(S) = \frac{f(1) - \Phi_V(N)}{\mu(I)} \mu(S) \quad \forall S \subset I/N, \) (2)

where \( N \) is the smallest carrier of \( \nu \).

Now let \( A \subset I \) be a measurable subset of \( I \). Separating the atoms in \( A \) we have:

\[
A = S \cup \bigcup_{s \in A \cap N} \{s\} \quad \text{where } S \subset I/N.
\]

Let us show that, for all admissible sequences \( \varphi = (\Pi_1, \Pi_2, \ldots) \) starting from \( \Pi_1 = (A, I/A) \):

\[
\varphi_{\Pi_i}(A_i) \rightarrow \Phi_V(A) \quad \text{at } i + \infty
\]

where \( A_i = \{j \in \Pi_i : j \subset A\} \) and \( \Phi_V(A) = \Phi_V(S) + \sum_{s \in A \cap N} \Phi_V(\{s\}) \).

From the additivity of the value it will be sufficient to prove (3) first for a set \( S \) lying in \( I/N \) and then for an atom \( s \in N \). So, let us take \( S \subset I/N \) measurable and

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3 Cf. theorem VI, 11 and 12 in Mood-Graybill-Boes (1974).
prove that for all admissible sequences \( \varphi = (\Pi_1, \Pi_2, \ldots) \) starting from \( \Pi_i = (S, I/S) \) we have:

\[
\varphi_{\Pi_i}(S_j) \rightarrow \Phi_\varphi(S) \quad (4)
\]

with \( S_j = \{ j \in \Pi_j : j \subset S \} \).

We shall suppose, without loss of generality that, for \( i \) sufficiently large, the atoms (i.e., the elements in \( N \)) are all separated in the partition \( \Pi_i \), and hence that: \( p_i = r_i + n \) where \( r_i \) denotes the number of members of \( \Pi_i \) in \( I/N \) and \( p_i \) the number of members of \( \Pi_i \).

\[
\varphi_{\Pi_i}(S_j) = \sum_{j \in S_j} \varphi_{\Pi_i}((j)).
\]

\[
\varphi_{\Pi_i}((j)) = \sum_{T \supset j} \frac{(p_i - t)! (t - 1)!}{p_i!} \left[ \nu_{\Pi_i}(T) - \nu_{\Pi_i}(T - \{j\}) \right] \nu_{\Pi_i}(T).
\]

where \( T \) denotes a coalition of \( t \) players of the game \( v_{\Pi_i} \).

We shall write: \( T = R \cup L \) where \( R \) consists of all players of \( v_{\Pi_i} \) in \( I/N \) and \( L \) contains atoms, with \( |R| = r \) and \( |L| = l \).

\[
\varphi_{\Pi_i}((j)) = \sum_{T \supset j} \frac{(p_i - t)! (t - 1)!}{p_i!} \left[ f(\mu(R) + \nu(L)) - f(\mu(R) + \nu(L) - \theta \mu(j)) \right]
\]

\[
= \sum_{L \subset N} \sum_{R \supset j} \frac{(r_i - r)! (r + l - 1)!}{r_i!} \mu(j) f_{\nu}(\mu(R) + \nu(L) - \theta \mu(j))
\]

\[
0 < \theta < 1,
\]

\[
= \mu(j) \sum_{L \subset N} \sum_{R \supset j} \left\{ \frac{(r_i - r)! (r + l - 1)!}{r_i!} \right\} g_L(\mu(R) - \theta \mu(j))
\]

\[
\times \frac{r_i! (p_i - r - l)! (r + l - 1)!}{p_i! (r_i - r)! (r - 1)!}.
\]

with

\[
g_L(x) = f(x + \nu(L)).
\]

Let

\[
K = \frac{r_i! (p_i - r - l)! (r + l - 1)!}{p_i! (r_i - r)! (r - 1)!}.
\]

\[
= \left( 1 - \frac{1}{p_i} \right) \frac{1}{p_i!} \ldots \left( 1 - \frac{n - 1}{p_i} \right) \sum_{a=1}^{l} \sum_{\beta=0}^{n-l} \frac{1}{p_i^{n-\alpha-\beta}} C_\alpha C_\beta \left( 1 - \frac{r}{p_i} \right)^{\beta} \left( \frac{r}{p_i} \right)^{\alpha}
\]

where \( C_\alpha \) and \( C_\beta \) only depend on \( l \) and \( n \).

\[
\Rightarrow \varphi_{\Pi_i}((j)) = \mu(j) \sum_{L \subset N} \sum_{a=1}^{l} \sum_{\beta=0}^{n-l} \left( 1 - \frac{1}{p_i} \right) \ldots \left( 1 - \frac{n - 1}{p_i} \right) p_i^{n-\alpha-\beta} \sum_{R \supset j} A_{\alpha \beta}(R)
\]

with

\[
A_{\alpha \beta}(R) = \frac{(r_i - r)! (r - 1)!}{r_i!} g_L(\mu(R) - \theta \mu(j)) \left( \frac{r}{p_i} \right)^{\alpha} \left( 1 - \frac{r}{p_i} \right)^{\beta}.
\]
We shall show that:

\[
\sum_{R \ni j} A_{\alpha \beta} (R) - \frac{1}{r_i \!-\! 1} \frac{1}{C'_{\tau_i -1}} g'_{\lambda} \left[ \frac{r - 1}{r_i - 1} \mu(I) \right] \left( \frac{r - 1}{r_i - 1} \right)^{\alpha} \left( 1 - \frac{r - 1}{r_i - 1} \right)^{\beta}
\]

converges to 0 uniformly in \( j \).

This is the same as showing that, given a partition \( \mu(I) = \sum_{j=1}^{\tau_i - 1} \alpha_j \) and given a continuous function \( g \) on \([0, \mu(I)]\),

\[
A = \sum_{R \ni j} \frac{(r_i - r)! (r - 1)!}{r_i !} \left[ g \left( \sum_{j \in R} \alpha_j \right) \left( \frac{r}{p_i} \right)^{\alpha} \left( 1 - \frac{r}{p_i} \right)^{\beta} - \frac{p_i - n}{p_i - n - 1} g \left( \frac{r - 1}{r_i - 1} \mu(I) \right) \left( \frac{r - 1}{r_i - 1} \right)^{\alpha} \left( 1 - \frac{r - 1}{r_i - 1} \right)^{\beta} \right]
\]

converges to 0 uniformly in \( j \), and this follows easily from the proof in Kannai (1966).

Using the definition of the Riemann integral:

\[
\int_0^1 g'_{\lambda} \left[ x \mu(I) \right] x^{\alpha} (1 - x)^{\beta} dx
\]

\[
= \lim_{i \uparrow + \infty} \sum_{r} \frac{1}{r_i - 1} g'_{\lambda} \left[ \frac{r - 1}{r_i - 1} \mu(I) \right] \left( \frac{r - 1}{r_i - 1} \right)^{\alpha} \left( 1 - \frac{r - 1}{r_i - 1} \right)^{\beta}
\]

\[
= \lim_{i \uparrow + \infty} \sum_{R \ni j} \frac{1}{(r_i - 1) C'_{\tau_i -1}} g'_{\lambda} \left[ \frac{r - 1}{r_i - 1} \mu(I) \right] \left( \frac{r - 1}{r_i - 1} \right)^{\alpha} \left( 1 - \frac{r - 1}{r_i - 1} \right)^{\beta}
\]

\[
= \sum_{R \ni j} A_{\alpha \beta} (R) \rightarrow \int_0^1 g'_{\lambda} \left[ x \mu(I) \right] x^{\alpha} (1 - x)^{\beta} dx \quad \text{uniformly in} \quad j,
\]

and hence,

\[
\lim \varphi_{\Pi} (\{ j \}) = \sum_{L \in N} \mu(j) \int_0^1 g'_{\lambda} \left[ x \mu(I) \right] x^{\prime} (1 - x)^{\alpha - \prime} dx.
\]

(All the terms in \( 1/p_i^{\alpha - \beta} \) have limit 0 except for \( n = \alpha + \beta \).)

\[
= \lim \varphi_{\Pi} (S_i) = \sum_{j \in S_i} \mu(j) \sum_{L \in N} \int_0^1 f' \left[ x \mu(I) + \nu(L) \right] x^{\prime} (1 - x)^{\alpha - \prime} dx
\]

\[
= \mu(T) \sum_{L \in N} \int_0^1 f' \left[ x \mu(I) + \nu(L) \right] x^{\prime} (1 - x)^{\alpha - \prime} dx
\]

\[
= \mu(T) \left\{ \left[ x^{\prime} (1 - x)^{\alpha - \prime} \frac{f' \left[ x \mu(I) + \nu(L) \right]}{\mu(I)} \right]_0^1 \right. \right.
\]

\[
\left. - \int_0^1 \frac{x^{\prime} (1 - x)^{\alpha - \prime - 1} (1 - nx)}{\mu(I)} f' \left[ x \mu(I) + \nu(L) \right] dx \right\},
\]
\[ \lim \varphi_{\Pi_i}(S_i) = \frac{\mu(T)}{\mu(I)} \left[ f(1) - \sum_{s \in N} \sum_{\pi \in \Pi} \int_0^1 \frac{(1 - x)^{n - s + 1} x^{\pi s - 1} (x - n)^{-1} \pi^{-1} (I_{\pi s})}{(\pi s)! (n - \pi s)!} \right] \times \int [x \mu(I) + \nu \circ \pi^{-1} (I_{\pi s})] \, dx \]

\[ = \frac{\mu(T)}{\mu(I)} \left[ f(1) - \sum_{s \in N} \sum_{\pi \in \Pi} \int_0^1 \frac{(1 - x)^{n - s} x^{s + 1} - 1 \pi^{-1} (I_{\pi s})}{(\pi s)! (n - \pi s - 1)!} \int [x \mu(I) + \nu \circ \pi^{-1} (I_{\pi s - 1})] \, dx \right] \]

\[ \lim \varphi_{\Pi_i}(S_i) = \frac{\mu(T)}{\mu(I)} \left[ f(1) - \sum_{s \in N} \sum_{\pi \in \Pi} \int_0^1 \frac{(1 - x)^{n - s} x^{s + 1}}{(\pi s)! (n - \pi s)!} \Delta(\pi s, x, \nu \circ \pi^{-1}, f) \, dx \right] \]

\[ = \frac{\mu(T)}{\mu(I)} \left[ f(1) - \sum_{s \in N} \Phi_{\nu} \{\{s\}\} \right] \]

\[ = \Phi_{\nu} (S_i) \]

and this proves the result (4) for all measurable \( S \subseteq I/N \).

Now let \( s \in N \) and \( \Pi_i = \{\{s\}, I/\{s\}\} \) be a partition of \( I \). Let us show that, for all admissible sequences: \( \varphi_{i} = (\Pi_1, \Pi_2, \ldots) \)

\[ \varphi_{\Pi_i}(\{s\}) \to \Phi_{\nu}(\{s\}) \] (5)

Using the same notation as before:

\[ \text{card } \Pi_i = p_i = r_i + n. \]

we have:

\[ \varphi_{\Pi_i}(\{s\}) = \sum_{T \supseteq \{s\}} \frac{(p_i - t)! (t - 1)!}{p_i!} \left[ \nu_{\Pi_i}(T) - \nu_{\Pi_i}(T - \{s\}) \right]. \]

Define \( T = R \cup L \) where \( R \) contains all players of \( \nu_{\Pi_i} \) in \( I/N \) and \( L \) consists of the atoms:

\[ \varphi_{\Pi_i}(\{s\}) = \sum_{L \supseteq \{s\}} \sum_{R \subseteq I/N} \frac{(p_i - t)! (t - 1)!}{p_i!} \left[ \nu_{\Pi_i}(T) - \nu_{\Pi_i}(T - \{s\}) \right] \]

\[ = \sum_{L \supseteq \{s\}} \sum_{R \subseteq I/N} \frac{(p_i - t)! (t - 1)! r_i!}{p_i! r_i! (r_i - t)!} \left[ \frac{r_i! (r_i - r)!}{r_i!} \left( g_L \left[ \mu(R) \right] - h_L \left[ \mu(R) \right] \right) \right] \]

with \( g_L(x) = f[x + \nu(L)], h_L(x) = f[x + \nu(L) - \nu(\{s\})] \); and by a proof as in (3) we have:

\[ \varphi_{\Pi_i}(\{s\}) = \sum_{L \supseteq \{s\}} \sum_{r = 0}^{r_i} \frac{1}{r_i} \left( 1 - \frac{r}{r_i} \right)^{n - r - 1} \left( \frac{r}{r_i} \right)^l g_L \left[ \frac{r}{r_i} \mu(I) \right] - h_L \left[ \frac{r}{r_i} \mu(I) \right] \]

converges to 0.
Hence, recalling the definition of the Riemann integral:

\[
\lim_{\ell \to \infty} \varphi_{\mu,\nu}(s) = \sum_{L \supset \{s\}} \int_0^1 (1 - x)^{n - l - 1} x^l \left\{ g_L[x\mu(I)] - h_L[x\mu(I)] \right\} dx \\
= \sum_{L \supset \{s\}} \int_0^1 \frac{x^l(1 - x)^{n - l - 1}}{l!(n - l - 1)!} \{ f[x\mu(I) + \nu(L)] - f[x\mu(I) + \nu(L) - \nu(\{s\})] \} dx \\
= \sum_{\pi \in \Pi} \int_0^1 \frac{x^{\pi s - 1}(1 - x)^{n - \pi s}}{(\pi s - 1)!(n - \pi s)!} \left\{ f[x\mu(I) + \nu \circ \pi^{-1} j_{w_s}] - f[x\mu(I) + \nu \circ \pi^{-1} j_{w_{s-1}}] \right\} dx \\
= \Phi v(\{s\}).
\]

This proves (5) and ends the proof of (3) and thus of our theorem.

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References


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