Calculus Review

1 Differentiation of one variable

**Definition:** The process of finding the derivative of a function is called differentiation.

**Notation:** For function $y = f(x)$, the first derivative is generally denoted as $f'(x)$ or $\frac{dy}{dx}$.

The derivative of $f$ at $x$ is given by:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (1)$$

1.1 Derivative as the slope of the tangent line

The slope of a straight line (linear function $y = f(x)$) is:

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x + \Delta x) - f(x)}{\Delta x}, \text{ if } x_2 - x_1 = \Delta x \quad (2)$$

If now we are working with a non-linear function such as $f(x) = x^2$, the slope of the curve changes at each point of the curve. To find a general formula of the slope, we use the derivative of the function $f(x)$. In our example, the slope of the curve $x^2$ is given by its derivative which is $2x$.

1.2 Derivative as the rate of change

Coming back to the linear function $f$, its slope measures how much $f(x)$ increases for each unit increase in $x$. Thus, it measures the *rate of change* of the function $f$. In equation 2, $\Delta x$ measures the change in $x$ and $f(x + \Delta x) - f(x)$ the change in $y$.

1.3 Differentiation Rules

Suppose that $k$ is an arbitrary constant and that $f, g$ are differentiable functions at $x = x_1$.

1.3.1 The Constant Rule

The derivative of a constant $k$ is zero.

$$\frac{d}{dx}[k] = 0 \quad (3)$$

1.3.2 The Power Rule

If $n$ is a rational number, a simple power rule is

$$\frac{d}{dx}[x^n] = nx^{n-1} \quad (4)$$

A more general rule is:

$$\frac{d}{dx}[f(x)^n] = n(f(x)^{n-1})f'(x) \quad (5)$$

Example: If $f(x) = 3x^3, f'(x) = 9x^2$
If $g(x) = (2x + 4)^2, g'(x) = 2(2x + 4) \cdot 2 = 4(2x + 4) = 8x + 16$
1.3.3 The Product Rule
\[
\frac{d}{dx}[f(x) \cdot g(x)] = f(x)g'(x) + f'(x)g(x)
\] (6)

**Example:** If \( f(x) = 3x^3 \) and \( l(x) = (x + 4) \), then \((f(x) \cdot l(x))' = 9x^2(x + 4) + 3x^3 = 12x^3 + 36x^2\)

1.3.4 The Quotient Rule
\[
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \text{ with } g(x) \neq 0
\] (7)

**Example:** If \( f(x) = 3x^3 \) and \( l(x) = (x + 4) \), then \( \left(\frac{f(x)}{l(x)}\right)' = \frac{9x^2(x+4)-3x^3}{(x+4)^2} = \frac{6x^3+36x^2}{(x+4)^2} \)

1.3.5 The Sum and Difference Rules
\[(f \pm g)'(x) = f'(x) \pm g'(x)\] (8)

**Example:** If \( n(x) = 3x^4 + x^5 \), \( n'(x) = 12x^3 + 5x^4 \)

1.3.6 The Chain Rule

Let’s define the function \( h \) as \( h(x) = g(f(x)) \) and \( i \) as \( i(x) = f(g(x)) \)

**Example:** if \( f(x) = x + 4 \) and \( g(x) = x^2 \), then \( h(x) = g(f(x)) = (x + 4)^2 \) and \( i(x) = f(g(x)) = x^2 + 4 \).

\[
\frac{d}{dx}[g(f(x))] = g'(f(x))f'(x)
\] (9)

**Example:** \( h'(x) = 2(x + 4) \times 1 = 2x + 8 \)

1.3.7 The Derivative of the Log Function

A simple version of this rule is:
\[
\frac{d}{dx} \ln x = \frac{1}{x}
\] (10)

The more general rule is:
\[
\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}
\] (11)

**Example:** If \( f(x) = 3x^3 \) then \( (\ln(3x^3))' = \frac{9x^2}{3x^3} = \frac{3}{x} \)

2 Partial Derivative

For \( z = f(x, y) \), the partial derivatives \( f_x \) and \( f_y \) are denoted by
\[
\frac{\partial}{\partial x}[f(x, y)] = f_x(x, y) = \frac{\partial z}{\partial x}
\] (12)

and
\[
\frac{\partial}{\partial y}[f(x, y)] = f_y(x, y) = \frac{\partial z}{\partial y}
\] (13)

**Example:** Let’s define \( f(x, y) = 3x - x^2y^2 + 2x^3y \), then, \( f_x(x, y) = 3 - 2xy^2 + 6x^2y \) and \( f_y(x, y) = -2x^2y + 2x^3 \)